On the Besicovitch-Stability of some Noisy Subshifts of Finite Type

Léo Gayral

18/03/2021, ALEA Days

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Université de Toulouse

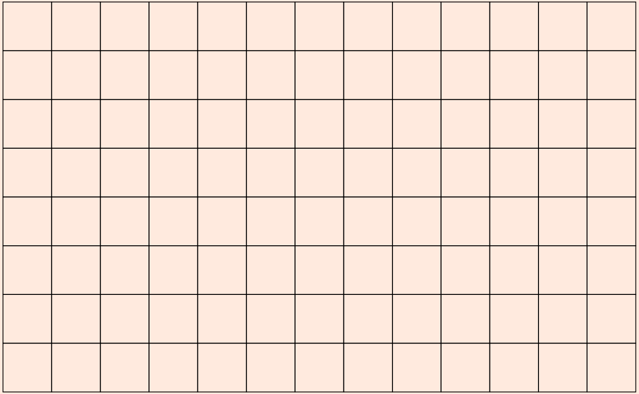
Crash Course on Noisy SFTs

Periodic Stability with Bernoulli Noises

Aperiodic Stability of a Robinson Tiling

Crash Course on Noisy SFTs

Subshifts of Finite Type



- Grid $\mathbb{Z}^2$.

Subshifts of Finite Type

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- Grid $\mathbb{Z}^2$.
- Alphabet $\mathcal{A} = \{\circ, \times\}$.

Subshifts of Finite Type

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- Grid $\mathbb{Z}^2$.
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- Forbidden patterns $\mathcal{F}$:

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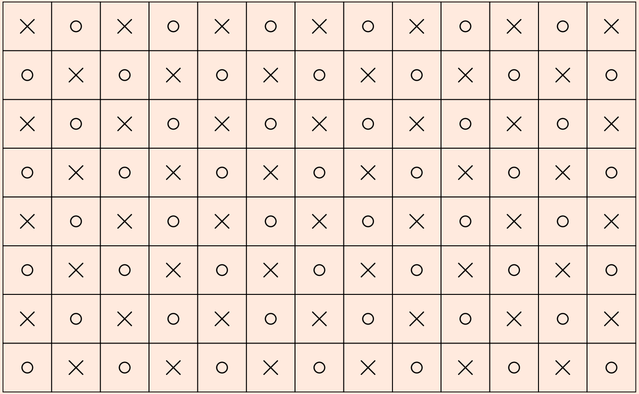
Subshifts of Finite Type



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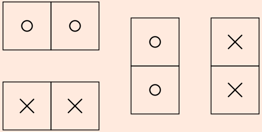

Subshifts of Finite Type

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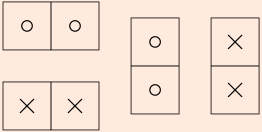

The SFT is the space $\Omega_{\mathcal{F}} \subset \mathcal{A}^{\mathbb{Z}^d}$ of such configurations.

Denote $\mathcal{M}_{\mathcal{F}}$ the σ-invariant measures on $\Omega_{\mathcal{F}}$.

Clair-Obscur Framework

- Inject $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}} = \mathcal{A} \times \{0,1\}$.

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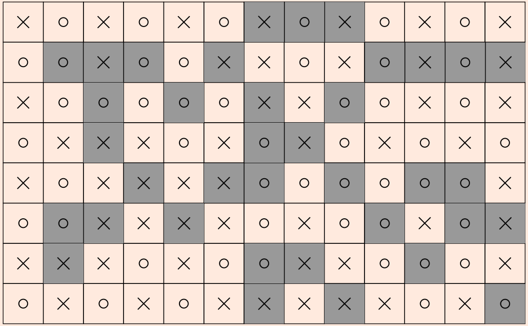

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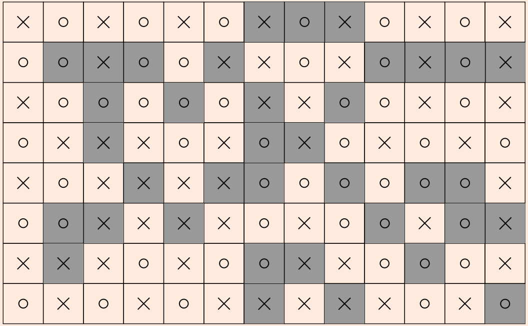
- Denote $\widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon) \subset \mathcal{M}_{\widetilde{\mathcal{F}}}$ the measures with $\mathcal{B}(\varepsilon)^{\otimes \mathbb{Z}^d}$ Bernoulli noise.



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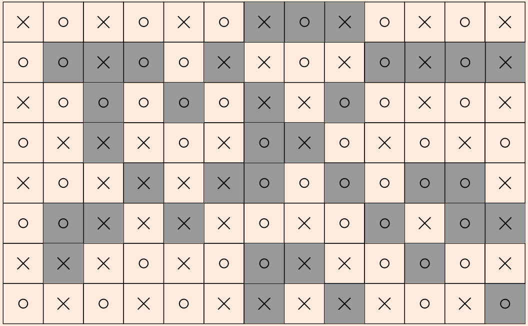
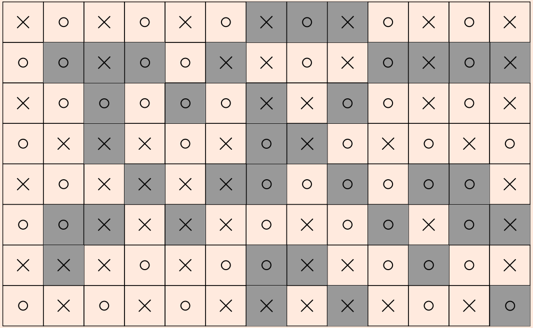
- Denote $\widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon) \subset \mathcal{M}_{\widetilde{\mathcal{F}}}$ the measures with $\mathcal{B}(\varepsilon)^{\otimes \mathbb{Z}^d}$ Bernoulli noise.
- The set $\widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon)$ is weak-* closed, and $\bigcap_{\varepsilon>0} \widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon) = \mathcal{M}_{\mathcal{F}}$.



Reminder (Weak-* Convergence)

We say that $\mu_n \xrightarrow{*} \mu$ *when* $\mu_n([w]) \to \mu([w])$ *for any finite pattern* w.

x



Finite Hamming distance:

$$d_{13\times 8}(x, \quad) = \frac{\quad}{13\times 8}$$

y

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Finite Hamming distance:

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Besicovitch Distance

$x|y$

×	○	×	○	×	○	×	○	×	○	×	○	×
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Finite Hamming distance:

$$d_{13\times 8}(x,y) = \frac{33}{13\times 8} \approx 0.3$$

Besicovitch Distance

$x|y$

×	○	×	○	×	○	×	○	×	○	×	○	×
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Hamming-Besicovitch pseudo-distance:

$$d_H = \limsup_{n\to\infty} d_{n\times n}$$

Besicovitch Distance

$x|y$

×	○	×	○	×	○	×	○	×	○	×	○	×
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Besicovitch distance on σ-invariant measures:

$$d_B(\mu,\nu) = \inf_{\lambda \text{ a coupling}} \int d_H(x,y)\mathrm{d}\lambda(x,y)$$

Stability

The SFT $\Omega_{\mathcal{F}}$ is f-stable for d_B on Bernoulli noises if:

$$\forall \varepsilon > 0, \quad \sup_{\lambda \in \widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon)} d_B\left(\pi_1^*(\lambda), \mathcal{M}_{\mathcal{F}}\right) \leq f(\varepsilon).$$

Stability

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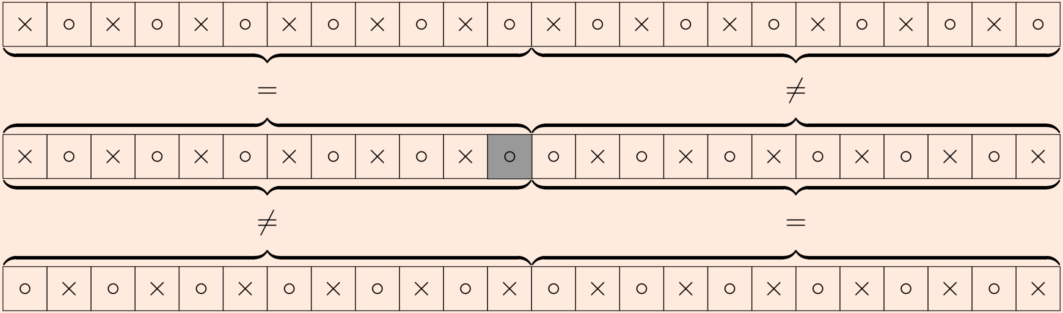
What kind of (in)stability results can we expect from typical SFTs ?

Periodic Stability
with Bernoulli Noises

1D Instability



Periodic Tilings

A SFT $\Omega_{\mathcal{F}}$ is (strongly) periodic if there exists an integer N such that any configuration is invariant for any translation in $(N\mathbb{Z})^d$.

Theorem

Consider $\Omega_{\mathcal{F}}$ a 2D+ periodic SFT.

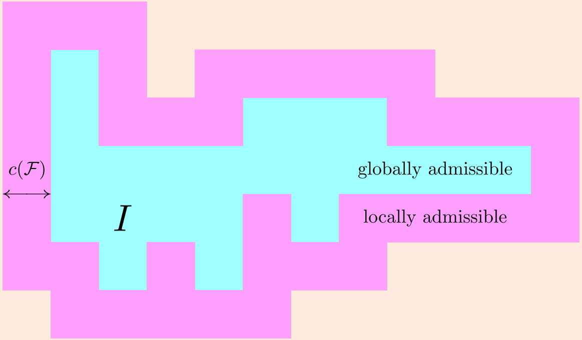
Then $\Omega_{\mathcal{F}}$ is f-stable on Bernoulli noises, with linear speed $f(\varepsilon) = 2C^d_{c(\mathcal{F})}\varepsilon$.

Reconstruction Function

Lemma

Consider a 2D+ periodic SFT $\Omega_{\mathcal{F}}$.

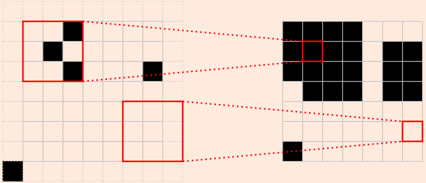
There exists $c(\mathcal{F}) \geq \lceil \frac{N}{2} \rceil$ such that, for any connected cell window $I \subset \mathbb{Z}^d$, if $w \in \mathcal{A}^{I+B_c}$ is locally admissible, then $w|_I$ is globally admissible.



Thickened Percolation

Consider $\varphi_n(b)_x = \max\limits_{\|y-x\|_\infty \le n} b_y$ for $b \in \{0,1\}^{\mathbb{Z}^d}$.

Starting from a site percolation ν, we obtain the n-thickened percolation $\varphi_n^*(\nu)$.



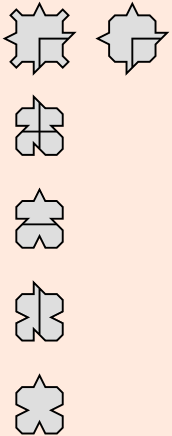
Proposition

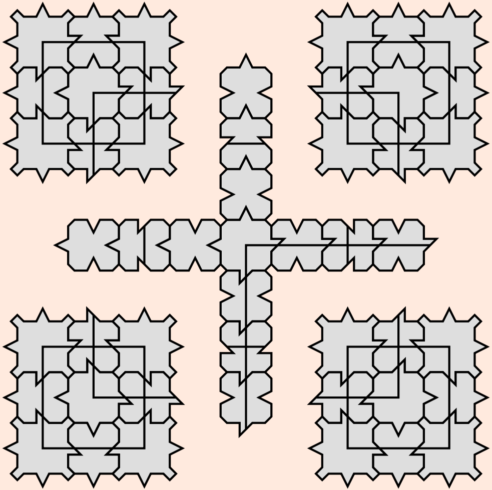
Consider $I \subset \mathbb{Z}^d$ the random infinite component of the n-thickened $\mathcal{B}(\varepsilon)^{\otimes \mathbb{Z}^d}$-percolation.

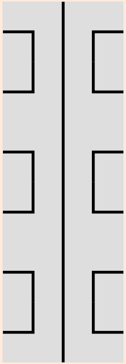
Then $C_n^d = 48(2n+1)^d$ is such that $\mathbb{P}(0 \notin I) \le C_n^d \times \varepsilon$.

Aperiodic Stability of a Robinson Tiling

The (Enhanced) Robinson Tiling







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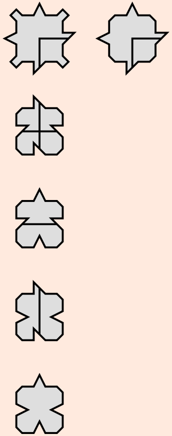

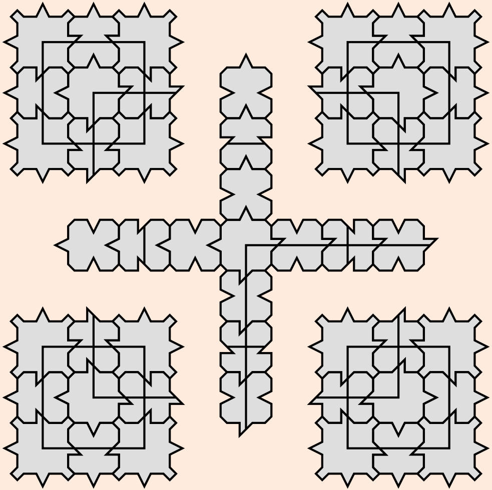

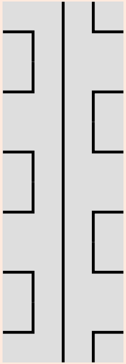

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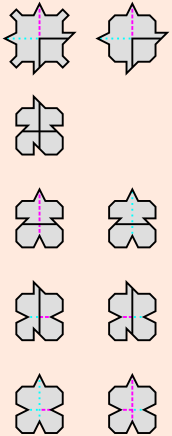
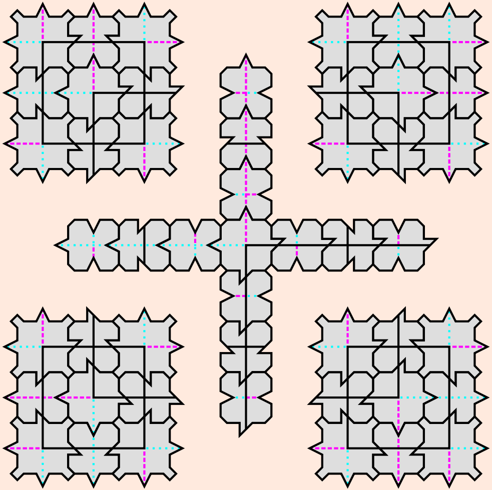
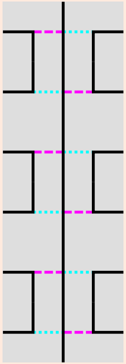


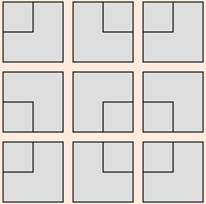




Reconstruction Function for the Enhanced Tiling

Proposition

For any scale $N \geq 2$, the constant $C_N = 2^N - 1$ is such that
for any integer n and any clear locally admissible pattern w on B_{n+C_N},
$w|_{B_n}$ is almost globally admissible, in the sense that up to a low-density grid,
$w|_{B_n}$ is made of well-aligned and well-oriented N-macro-tiles.



Density of the Grid

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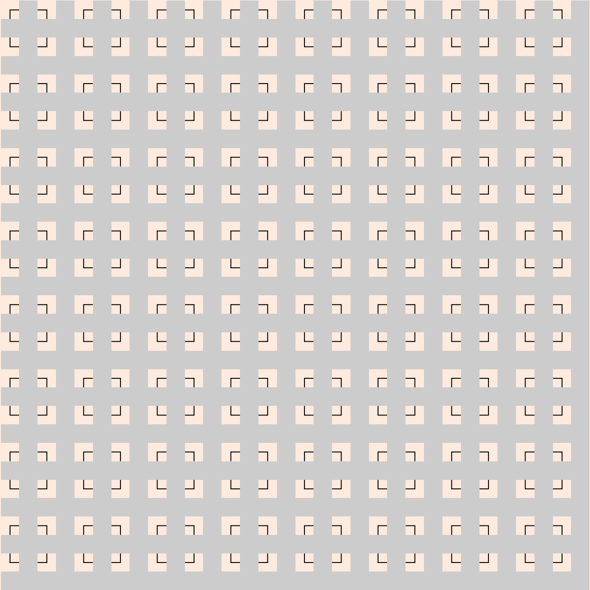

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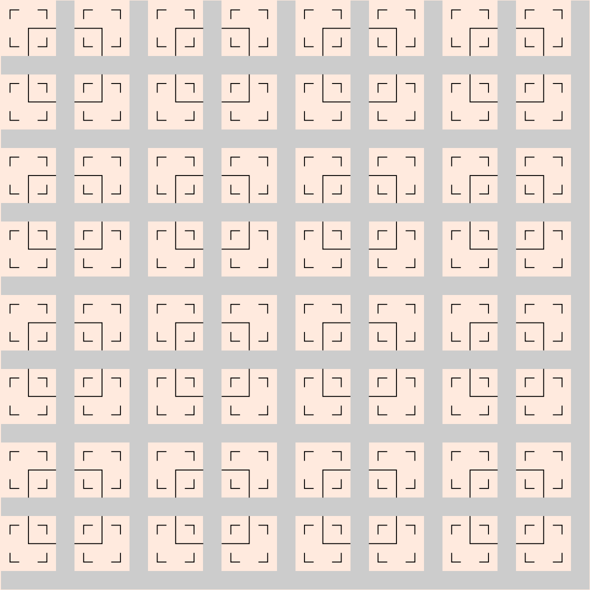

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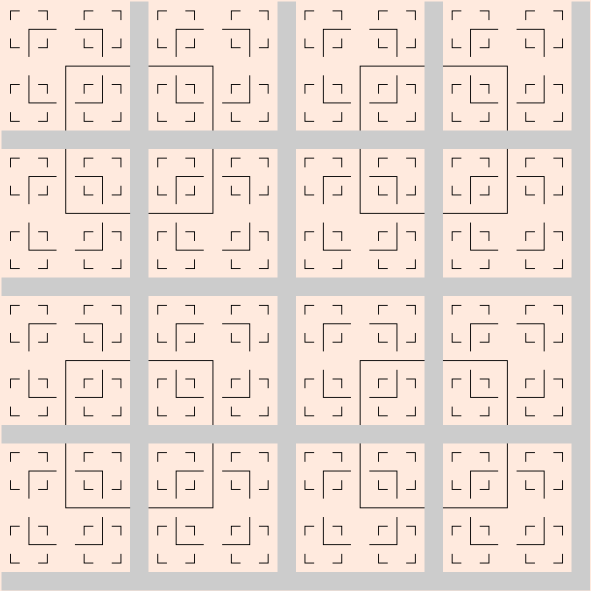

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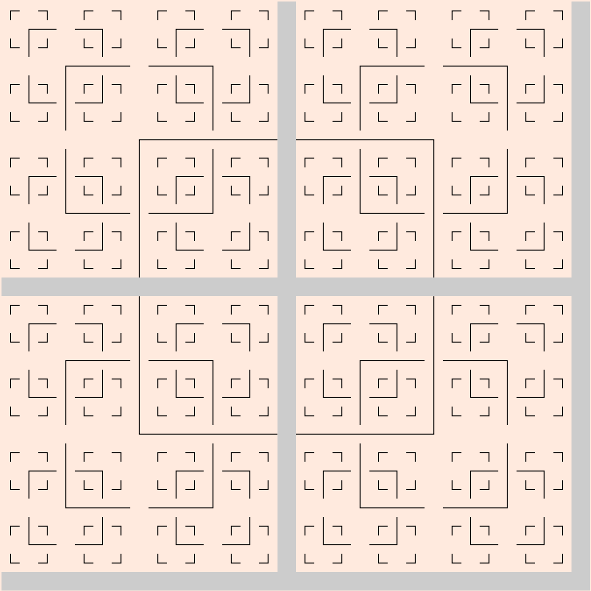

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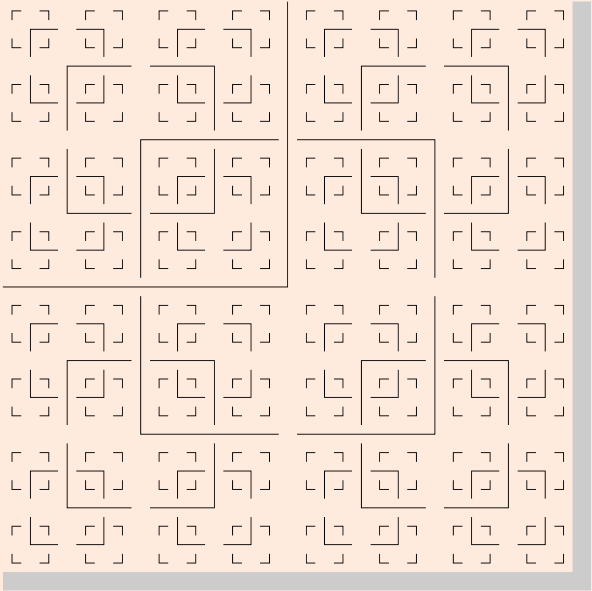

Non-linear Stability

Theorem

For any $\varepsilon > 0$, *any scale N, and any measure* $\mu = \pi_1^*(\lambda)$ *with* $\lambda \in \widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon)$:

$$d_B\left(\mu, \mathcal{M}_{\mathcal{F}}\right) \leq 96\left(2^{N+2}+1\right)^2 \varepsilon + \frac{1}{2^{N-1}}.$$

Hence, the SFT is f-stable with $f(\varepsilon) = 48\sqrt[3]{6\varepsilon}$.

Bibliography

Are there any questions?

Léo Gayral and Mathieu Sablik.
On the Besicovitch-stability of noisy random tilings.
https://arxiv.org/abs/2104.09885, 2021.