The Besicovitch-Stability of Noisy Tilings is Undecidable

Léo Gayral

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IMT, Université Toulouse III Paul Sabatier

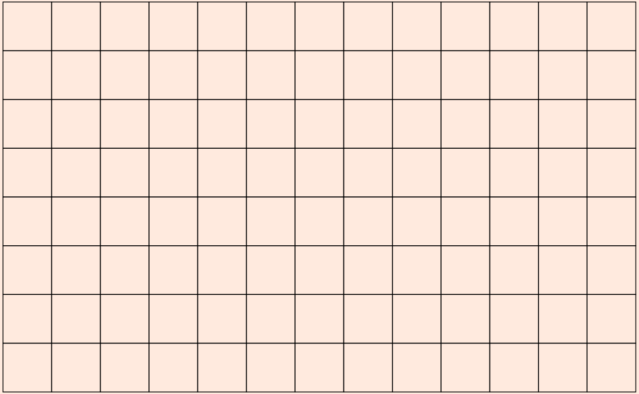
Crash Course on Noisy SFTs

Is the Robinson Tiling Stable ?

Undecidability Through the Simulation of Turing Machines

Crash Course on Noisy SFTs

Subshifts of Finite Type



- Grid $\mathbb{Z}^2$.

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- Grid $\mathbb{Z}^2$.
- Alphabet $\mathcal{A} = \{\circ, \times\}$.

Subshifts of Finite Type

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- Grid $\mathbb{Z}^2$.
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- Forbidden patterns $\mathcal{F}$:

○	○

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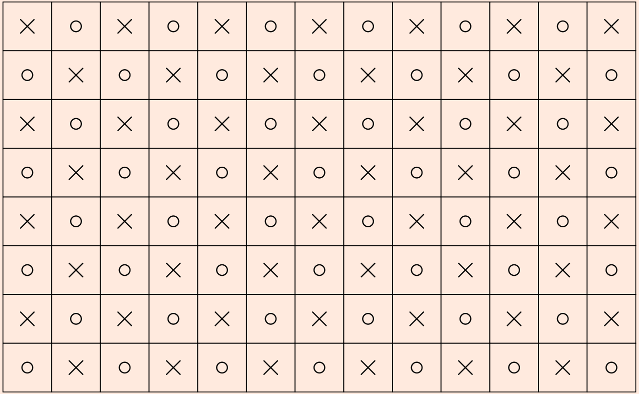
Subshifts of Finite Type



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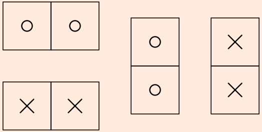

Subshifts of Finite Type

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- Forbidden patterns $\mathcal{F}$:

○	○

×	×

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×
×

The SFT is the space $\Omega_{\mathcal{F}} \subset \mathcal{A}^{\mathbb{Z}^d}$ of such configurations.

Denote $\mathcal{M}_{\mathcal{F}}$ the σ-invariant measures on $\Omega_{\mathcal{F}}$.

Clair-Obscur Framework

- Inject $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}} = \mathcal{A} \times \{0, 1\}$.

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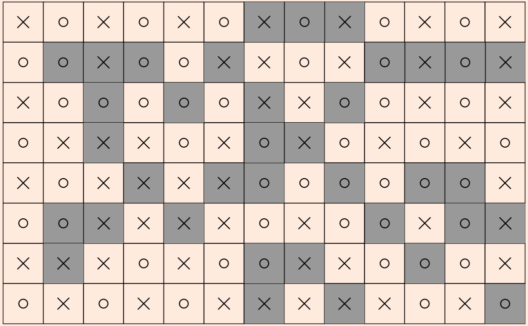
Clair-Obscur Framework

- Inject $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}} = \mathcal{A} \times \{0, 1\}$.
- Identify $\mathcal{F} \cong \widetilde{\mathcal{F}} = \mathcal{F} \times \{0\}$.



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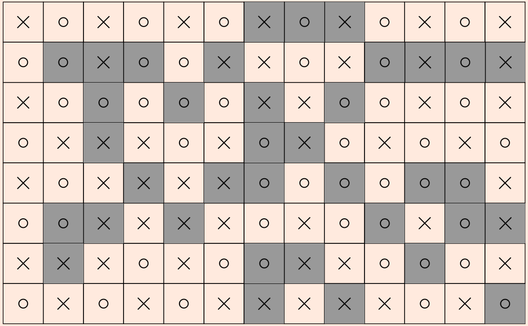
- Denote $\widetilde{\mathcal{M}^{\mathcal{B}}_{\mathcal{F}}}(\varepsilon) \subset \mathcal{M}_{\widetilde{\mathcal{F}}}$ the measures with $\mathcal{B}(\varepsilon)^{\otimes \mathbb{Z}^d}$ Bernoulli noise.



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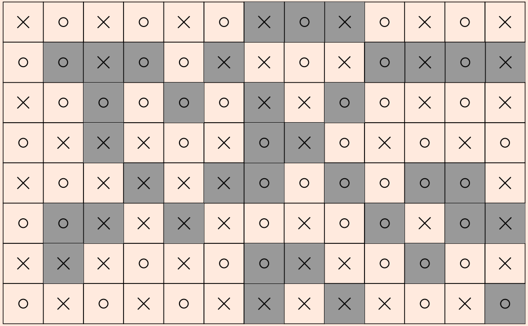
- Denote $\widetilde{\mathcal{M}^{\mathcal{B}}_{\mathcal{F}}}(\varepsilon) \subset \mathcal{M}_{\widetilde{\mathcal{F}}}$ the measures with $\mathcal{B}(\varepsilon)^{\otimes \mathbb{Z}^d}$ Bernoulli noise.
- The set $\widetilde{\mathcal{M}^{\mathcal{B}}_{\mathcal{F}}}(\varepsilon)$ is weak-* closed, and $\bigcap_{\varepsilon>0} \widetilde{\mathcal{M}^{\mathcal{B}}_{\mathcal{F}}}(\varepsilon) = \mathcal{M}_{\mathcal{F}}$.

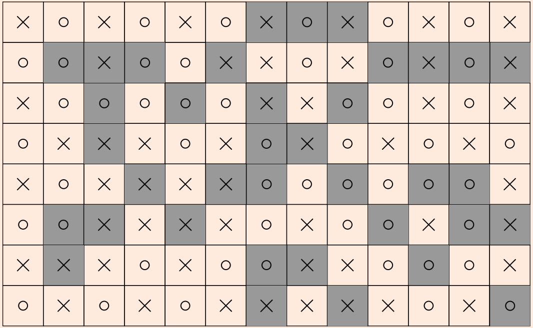


Reminder (Weak-* Convergence)

We say that $\mu_n \xrightarrow{*} \mu$ when $\mu_n([w]) \to \mu([w])$ for any finite pattern w.

Besicovitch Distance

x



Finite Hamming distance:

$$d_{13\times 8}(x, \quad) = \frac{\quad}{13\times 8}$$

Besicovitch Distance

y

×	○	×	○	×	○	×	○	×	○	×	○	×
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Besicovitch Distance

$x|y$

$\times$	$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\times$
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$\times$	$\circ$	$\otimes$	$\circ$	$\otimes$	$\circ$	$\times$	$\otimes$	$\otimes$	$\circ$	$\times$	$\circ$	$\times$
$\circ$	$\times$	$\otimes$	$\times$	$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\times$	$\circ$
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$\circ$	$\otimes$	$\otimes$	$\times$	$\otimes$	$\times$	$\circ$	$\times$	$\circ$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
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$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\times$	$\otimes$	$\times$	$\otimes$	$\times$	$\circ$	$\times$	$\circ$

Finite Hamming distance:

$$d_{13\times 8}(x,y) = \frac{33}{13\times 8} \approx 0.3$$

Besicovitch Distance

$x|y$

×	○	×	○	×	○	×	○	×	○	×	○	×
○	⊗	⊗	⊗	○	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗
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Hamming-Besicovitch pseudo-distance:

$$d_H = \limsup_{n\to\infty} d_{n\times n}$$

Besicovitch Distance

$x|y$

×	○	×	○	×	○	×	○	×	○	×	○	×
○	⊗	⊗	⊗	○	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗
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Besicovitch distance on σ-invariant measures:

$$d_B(\mu,\nu) = \inf_{\lambda \text{ a coupling}} \int d_H(x,y)\mathrm{d}\lambda(x,y)$$

Stability

The SFT $\Omega_{\mathcal{F}}$ is f-stable for d_B on Bernoulli noises if:

$$\forall \varepsilon > 0, \quad \sup_{\lambda \in \widetilde{\mathcal{M}^{\mathcal{B}}_{\mathcal{F}}}(\varepsilon)} d_B\left(\pi_1^*(\lambda), \mathcal{M}_{\mathcal{F}}\right) \leq f(\varepsilon).$$

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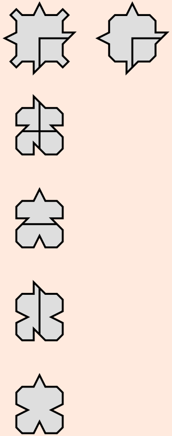
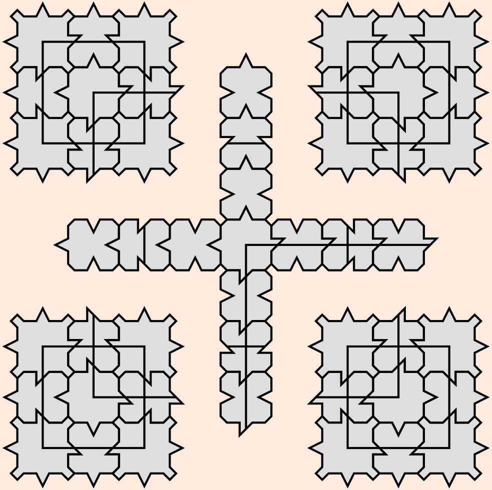
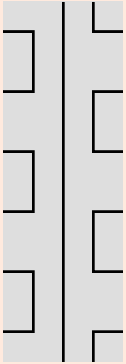
What kind of (in)stability results can we expect from typical SFTs ?

Is the Robinson Tiling Stable ?

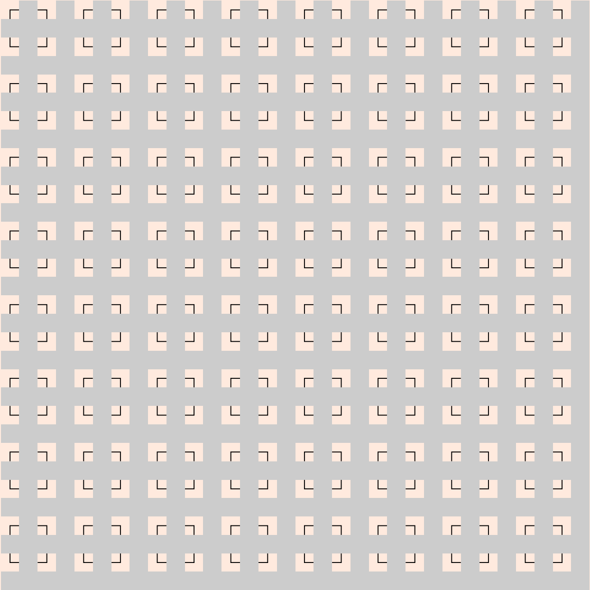
The Robinson Tiling







Hierarchical Structure of the Robinson Tiling



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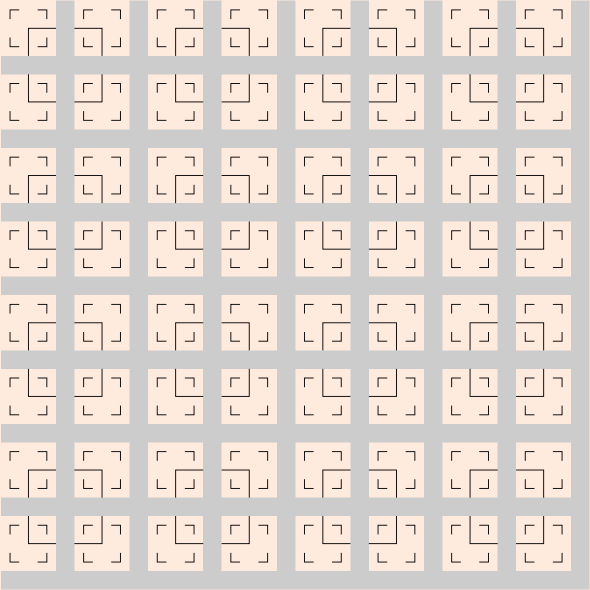

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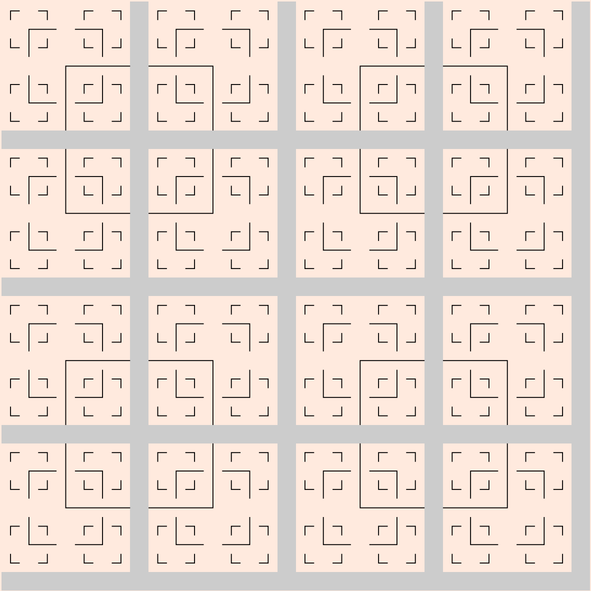

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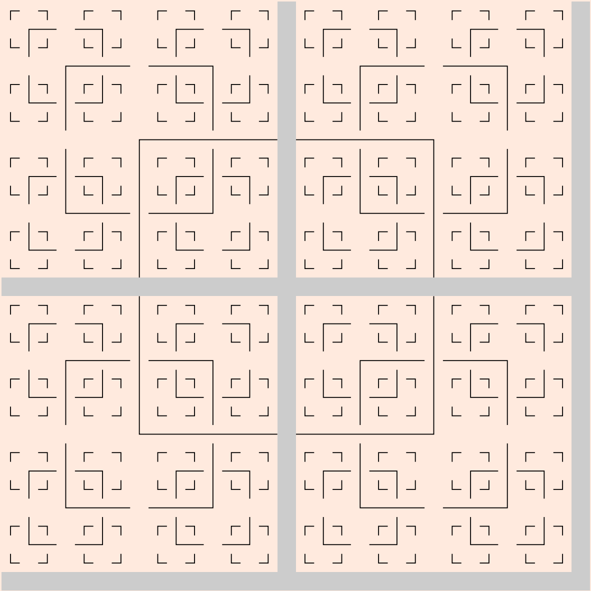
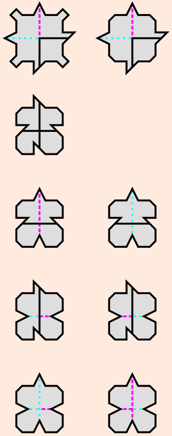
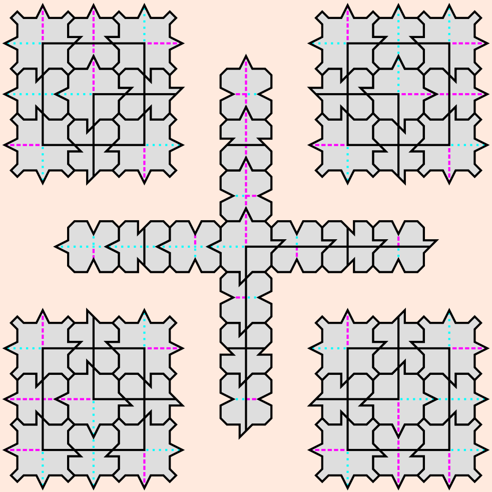
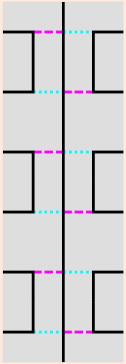


An Enhanced Robinson Tiling







Stability of the Enhanced Tiling

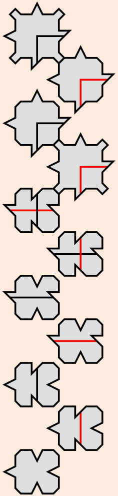
Theorem ([Gayral and Sablik, 2021, Proposition 7.8 and Theorem 7.9])

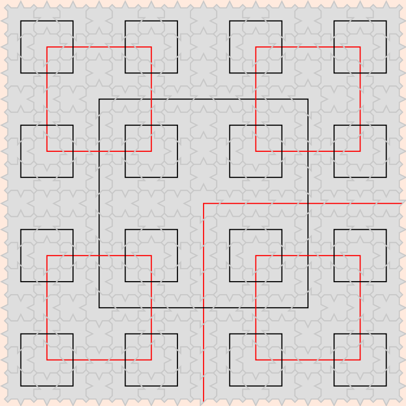
For any $\varepsilon > 0$, any scale N, and any measure $\mu = \pi_1^*(\lambda)$ with $\lambda \in \widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon)$:

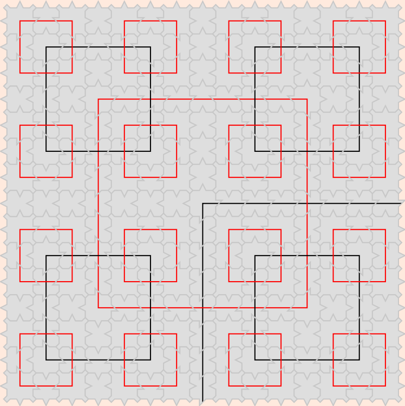
$$d_B\left(\mu, \mathcal{M}_{\mathcal{F}}\right) \leq 96\left(2^{N+2}+1\right)^2 \varepsilon + \frac{1}{2^{N-1}}.$$

Hence, the SFT is f-stable with $f(\varepsilon) = 48\sqrt[3]{6\varepsilon}$.

A Two-Coloured Robinson Tiling





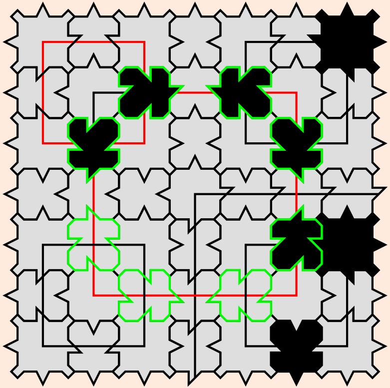


Unstability of the Red-Black Tiling

Proposition ([Gayral, 2021, Proposition 1])

The SFT Ω_{RB} is unstable.

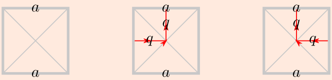
More precisely, for any $\varepsilon > 0$, we have $\mu \in \mathcal{M}^{\mathcal{B}}_{RB}(\varepsilon)$ such that $d_B(\mu, \mathcal{M}_{RB}) \geq \frac{1}{8}$.



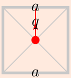
Undecidability Through the Simulation of Turing Machines

Consider a Turing machine $(Q, \Gamma, I, F, \delta)$ and define the following Wang tiles:

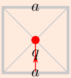
- For any letter $a \in \Gamma$ and any state $q \in Q$:



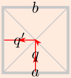
- For any letter $a \in \Gamma$ and initial state $q \in I$:



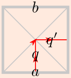
- For any letter $a \in \Gamma$ and final state $q \in F$:



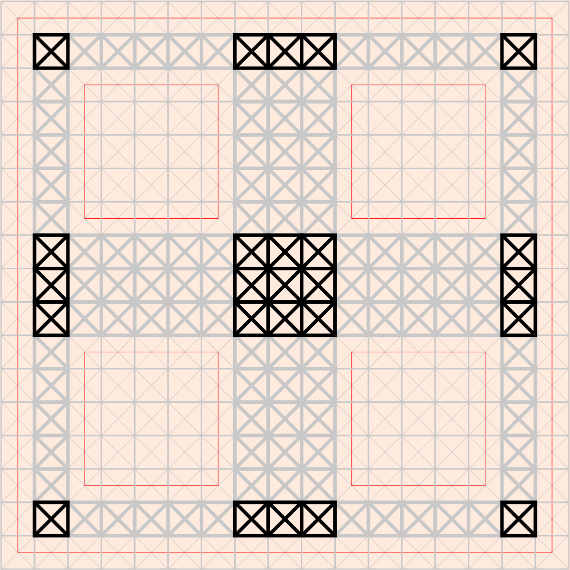
- For any transition $\delta(a, q) = (b, q', L)$:



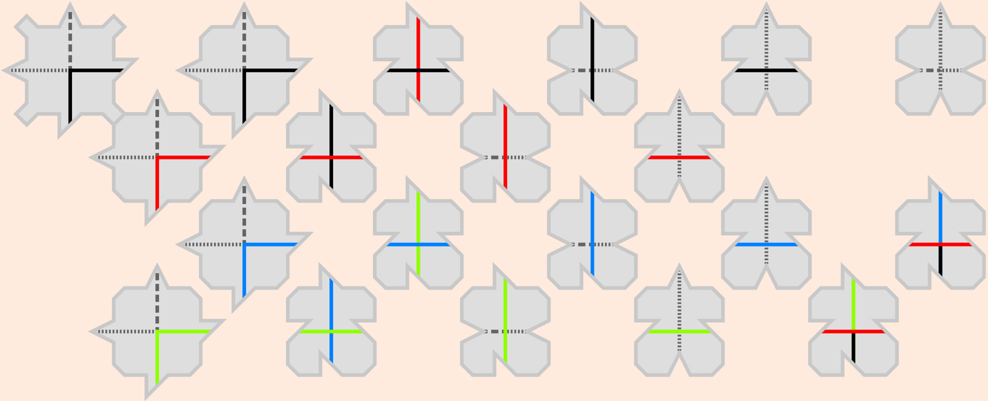
- For any transition $\delta(a, q) = (b, q', R)$:



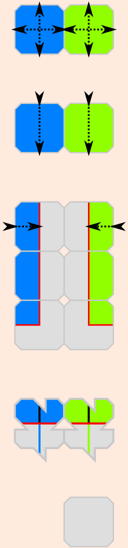
Embedding Space-Time Diagrams into Robinson Tilings

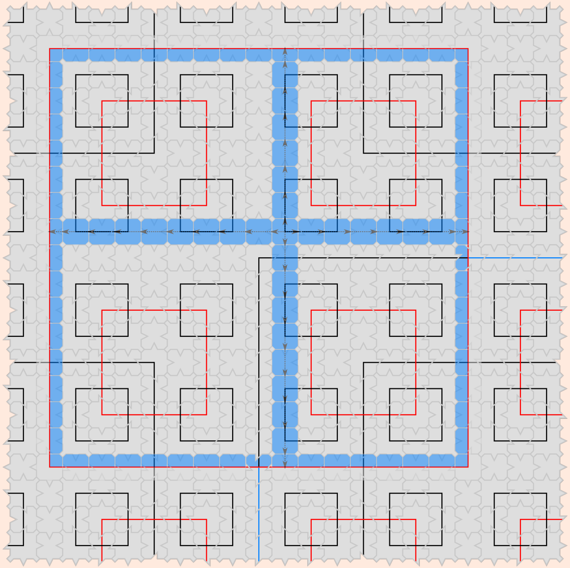


A Four-Coloured Enhanced Robinson Tiling



Transition from the Red-Black to the Blue-Green Phase





Undecidability of the Stability

Theorem ([Gayral, 2021, Theorem 1])

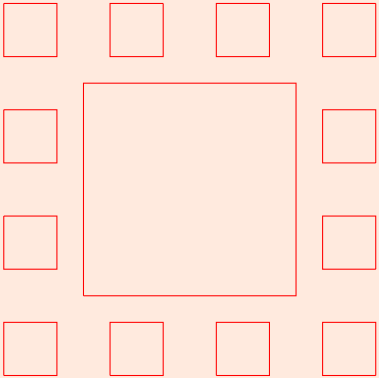
Let us denote by $\Omega_{\mathcal{T}}$ the SFT that embeds the Turing machine T into a variant of the Robinson tiling.

Then $\Omega_{\mathcal{T}}$ is stable (for d_B on the class $\mathcal{B}$) if and only if the Turing machine T does not end on the empty input. In the stable case, $\Omega_{\mathcal{T}}$ is polynomially stable.

Corollary ([Gayral, 2021, Corollary 1])

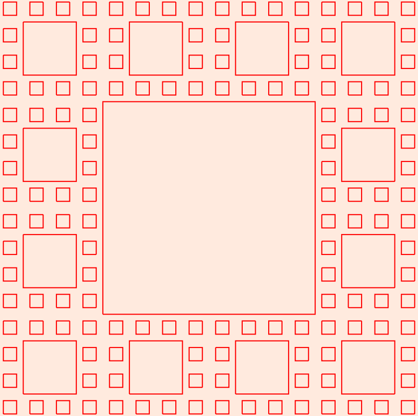
The problem of deciding whether the SFT $\Omega_{\mathcal{F}}$ is stable or not given the set of forbidden patterns $\mathcal{F}$ is undecidable.

The Stable Case



In a $2N$-macro-tile, only $O\left(12^N\right)$ tiles out of 16^N are ignored.

The Stable Case



In a $2N$-macro-tile, only $O\left(12^N\right)$ tiles out of 16^N are ignored.

The Unstable Case

We can do the same Blue-Green colour flip as in our Red-Black unstable example.

If the Turing machine stops in $2N$-macro-tiles,
we have a $\Omega\left(\frac{1}{16^N}\right)$ density of differences between Blue and Green.

Bibliography

Are there any questions?

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