From Noisy Tilings to Computable Analysis

How to fit the stability of noisy tilings into the arithmetical hierarchy?

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Joint work with Mathieu Sablik
IMT, Université Toulouse III Paul Sabatier

Framework for Noisy Tilings

Subshifts of Finite Type and Forbidden Patterns

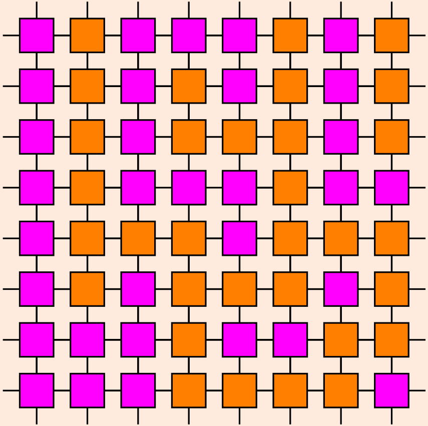

Figure 1: Example of configuration,

- Group $G = \mathbb{Z}^2$ with 2 generators.
- Alphabet $\mathcal{A} = \{$, $\}$.

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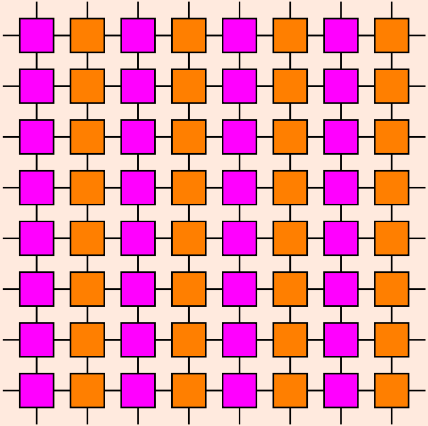

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- Group $G = \mathbb{Z}^2$ with 2 generators.
- Alphabet $\mathcal{A} = \{\square, \square\}$.
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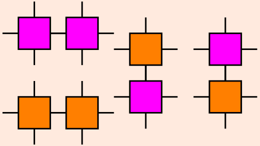

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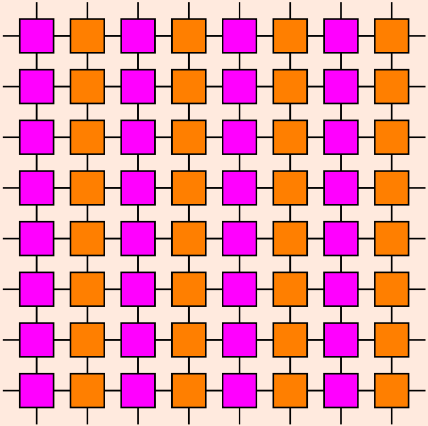

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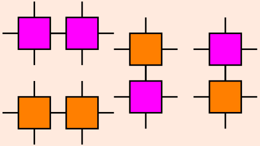

- The SFT is the space $\Omega_{\mathcal{F}} \subset \mathcal{A}^G$ of such configurations.

Example of Tiling

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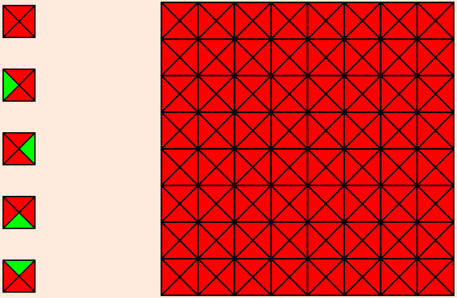

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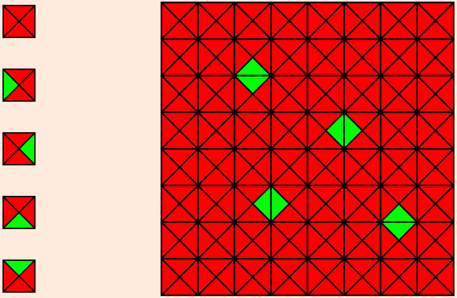

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Space of Measures

Denote $\mathcal{M}_{\mathcal{F}}$ the σ-invariant measures on $\Omega_{\mathcal{F}}$, such that $\sigma_k^*(\mu) = \mu$ for any k.

There are several ways of adding noise to tilings.

- Statistical Physics Viewpoint: Gibbs Measures

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- Statistical Physics Viewpoint: Gibbs Measures
- Information Theory Viewpoint: Bernoulli Noise

Clair-Obscur Framework

- Inject $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}} = \mathcal{A} \times \{0,1\}$.
- Identify $\mathcal{F} \cong \widetilde{\mathcal{F}} = \mathcal{F} \times \{0\}$.
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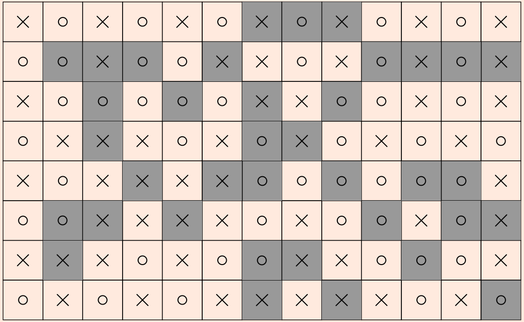

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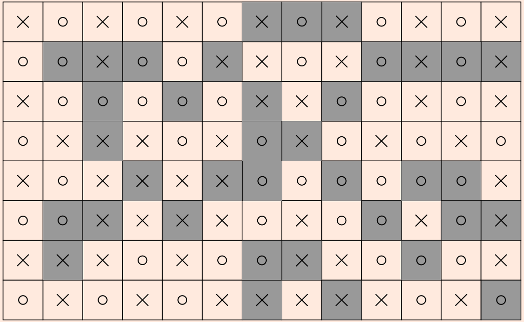

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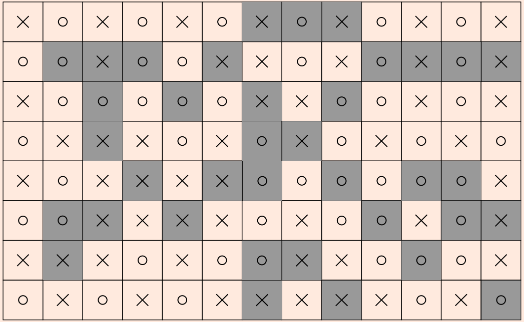

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Reminder (Weak-* Convergence)

We say that $\mu_n \xrightarrow{*} \mu$ *when* $\mu_n([w]) \to \mu([w])$ *for any finite pattern* w*.*

Besicovitch Distance

x

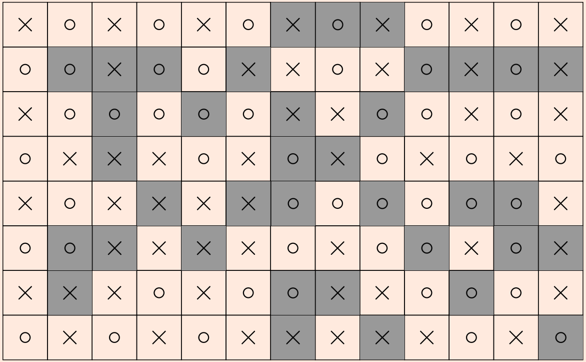

Figure 4: Frequency of differences between x and y.

Finite Hamming distance:

$d_{13\times 8}(x, \quad) = \frac{}{13\times 8}$

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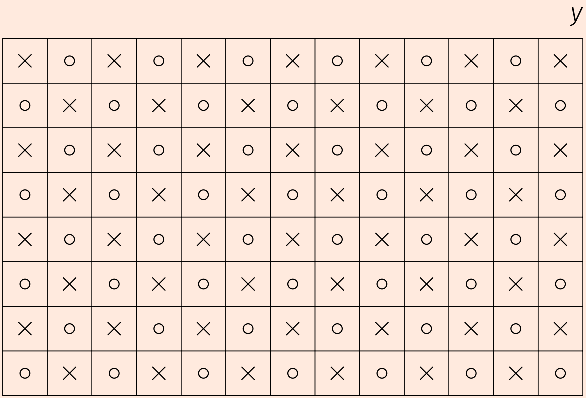

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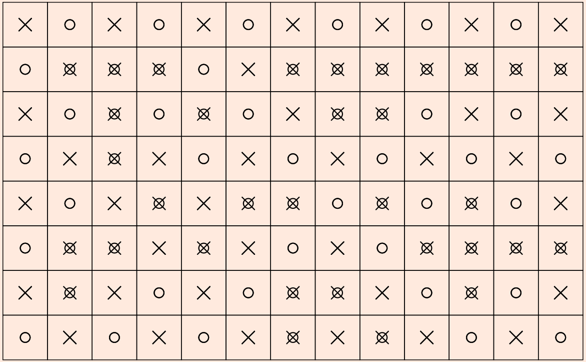

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Besicovitch distance on σ-invariant measures:

$$d_B(\mu,\nu) := \inf_{\lambda \text{ a coupling}} \int d_H(x,y)\mathrm{d}\lambda(x,y) = \inf_{\lambda \text{ a coupling}} \lambda\left([x_0 \neq y_0]\right)$$

Stability

Definition (Speed of Stability)

Let f s.t. $\lim_{x \to 0^+} f(x) = 0$. The SFT $\Omega_{\mathcal{F}}$ is f-stable for d_B on Bernoulli noises if:

$$\forall \varepsilon > 0, \quad \sup_{\lambda \in \widetilde{\mathcal{M}^{\mathcal{B}}_{\mathcal{F}}}(\varepsilon)} d_B\left(\pi_1^*(\lambda), \mathcal{M}_{\mathcal{F}}\right) \leq f(\varepsilon).$$

Informally, a generic ε-noisy configuration will be at distance $f(\varepsilon)$ of a tiling in $\Omega_{\mathcal{F}}$.

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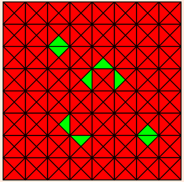

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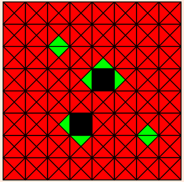

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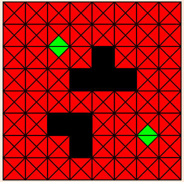

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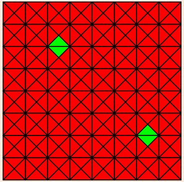

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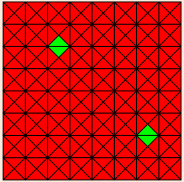

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Hence, this example is 5ε-stable.

Conjugacy Invariance

Theorem [Gayral and Sablik, 2021, Corollary 3.15]

Let $f : \Omega_{\mathcal{F}} \to \Omega_{\mathcal{F}'}$ be a conjugacy, a bi-continuous bijection, such that, for any $k \in \mathbb{Z}^d$, we have $\sigma_k \circ f = f \circ \sigma_k$.

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A fixed-point argument [Durand et al., 2012] already gave a stable aperiodic example.

Stability for Periodic Tilings

Periodic SFT

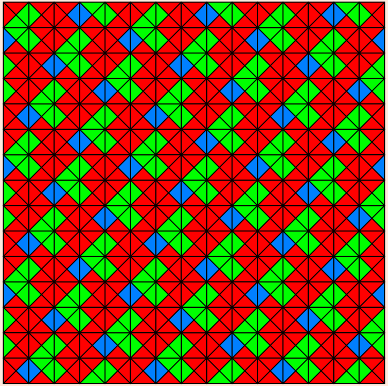

Figure 6: A periodic configuration,

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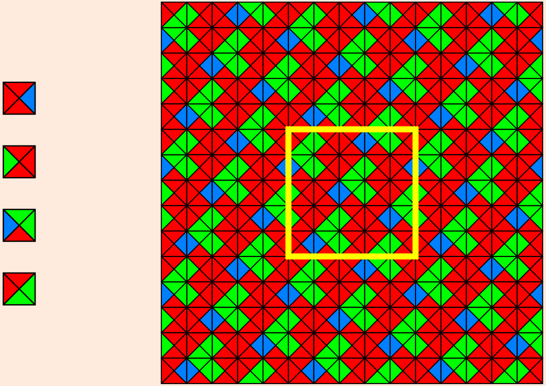
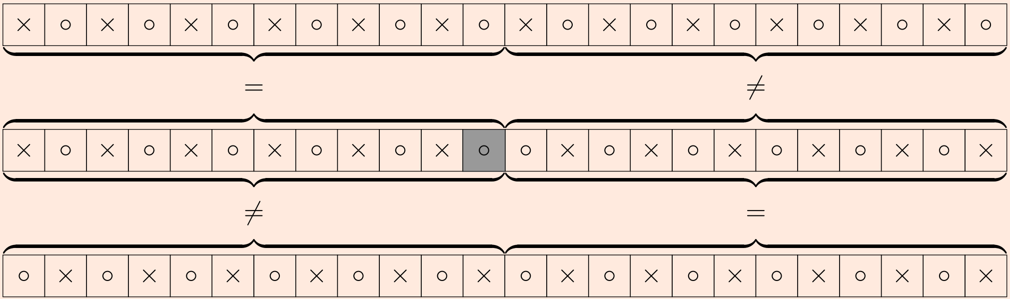


Figure 6: A periodic configuration, characterised by a base hypercube that repeats in all directions.

1D Classification of the Stability



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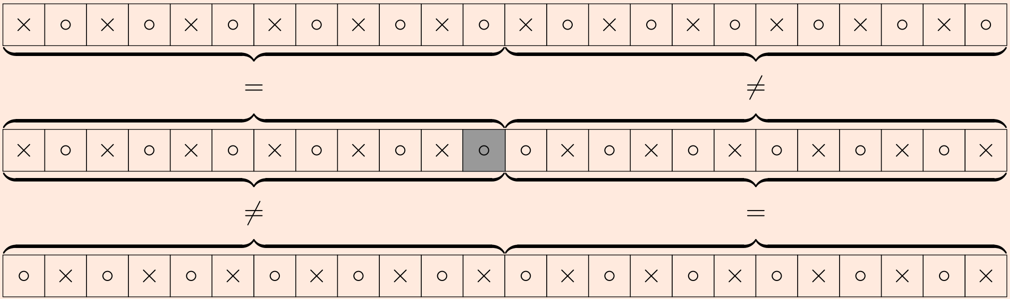

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Theorem [Gayral and Sablik, 2021, Theorem 4.8 and Theorem 4.9]

Consider $\Omega_{\mathcal{F}}$ a 1D SFT. Then $\Omega_{\mathcal{F}}$ is (linearly) stable on Bernoulli noises iff it is mixing.

Most notably, p-periodic SFTs (with $p \geq 2$) are unstable.

Periodic Tilings in Higher Dimensions

A SFT $\Omega_{\mathcal{F}}$ is (strongly) periodic if there exists an integer N such that any configuration is invariant for any translation in $(N\mathbb{Z})^d$.

Theorem [Gayral and Sablik, 2021, Theorem 5.7]

Consider $\Omega_{\mathcal{F}}$ a 2D+ periodic SFT.

Then $\Omega_{\mathcal{F}}$ is f-stable on Bernoulli noises, with linear speed $f(\varepsilon) = 2C^d_{c(\mathcal{F})}\varepsilon$.

Reconstruction Function

Lemma [Gayral and Sablik, 2021, Lemma 5.3]

Consider a 2D+ periodic SFT $\Omega_{\mathcal{F}}$.

There exists $c(\mathcal{F}) \geq \lceil \frac{N}{2} \rceil$ such that, for any connected cell window $I \subset \mathbb{Z}^d$, if $w \in \mathcal{A}^{I+B_c}$ is locally admissible, then $w|_I$ is globally admissible.

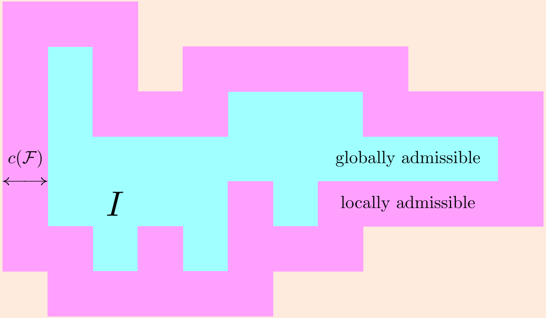

Figure 8: Here, the whole domain contains no forbidden pattern, but only the blue zone is guaranteed to be the restriction of an actual configuration.

Thickened Percolation

Consider $\varphi_n(b)_x = \max_{\|y-x\|_\infty \leq n} b_y$ for $b \in \{0,1\}^{\mathbb{Z}^d}$.

Starting from a site percolation ν, we obtain the n-thickened percolation $\varphi_n^*(\nu)$.

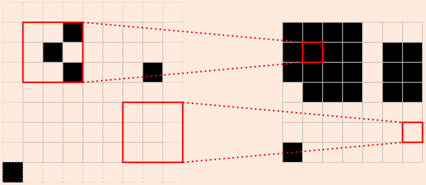

Figure 9: Illustration of the mapping φ_1.

Proposition [Gayral and Sablik, 2021, Proposition 5.6]

Consider $I \subset \mathbb{Z}^d$ the random infinite component of the n-thickened $\mathcal{B}(\varepsilon)^{\otimes \mathbb{Z}^d}$-percolation.

Then $C_n^d = 48(2n+1)^d$ is such that $\mathbb{P}(0 \notin I) \leq C_n^d \times \varepsilon$.

Stability for Aperiodic Tilings

The Robinson Tiling

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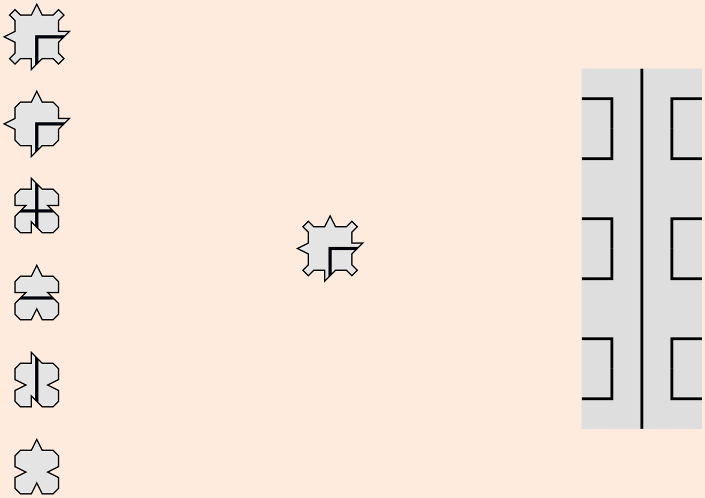
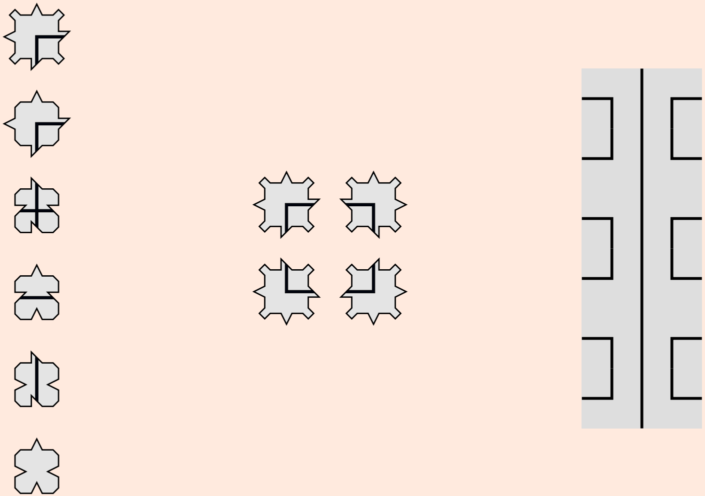


Figure 10: Hierarchical structure of the Robinson tiling.

The Robinson Tiling



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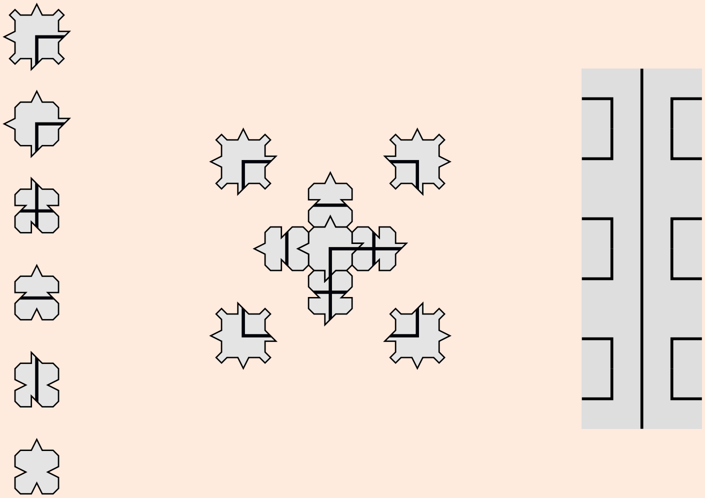

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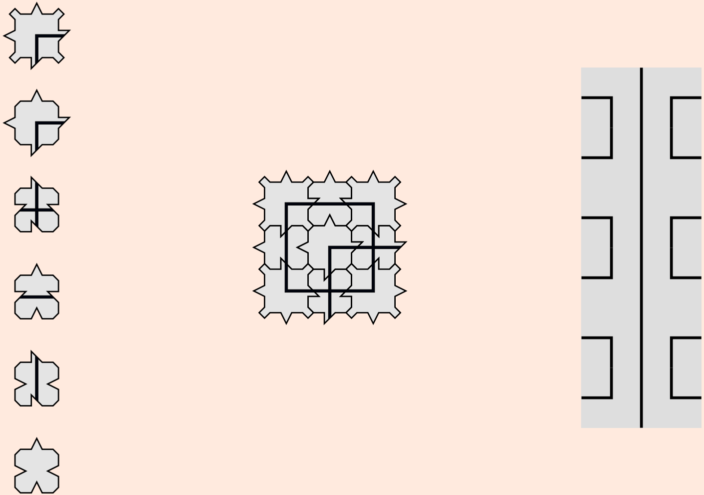

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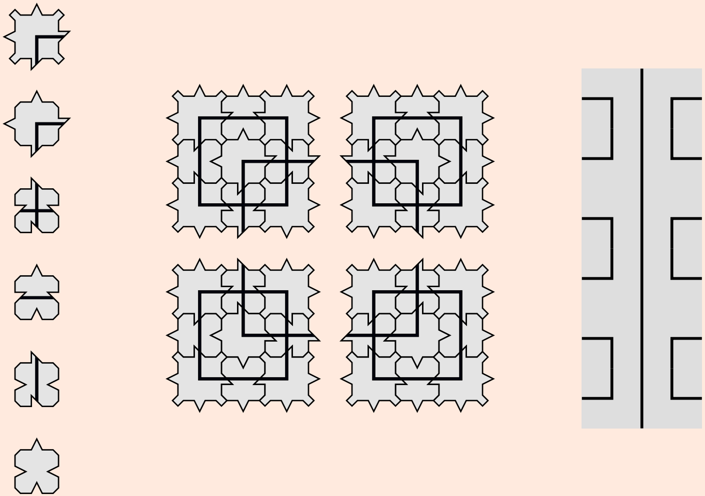

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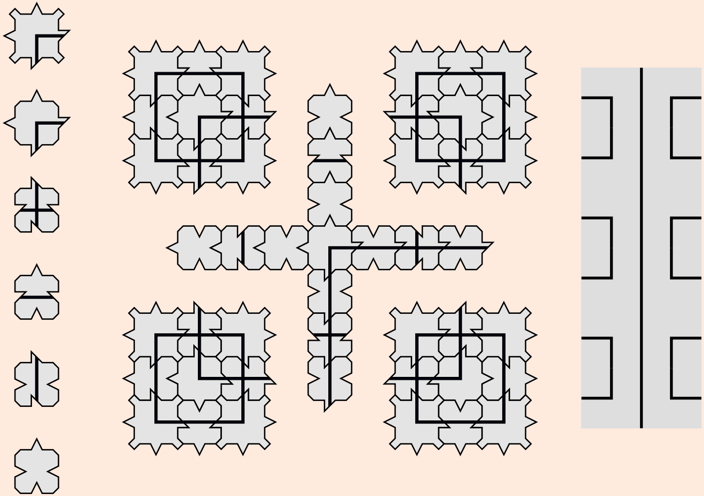

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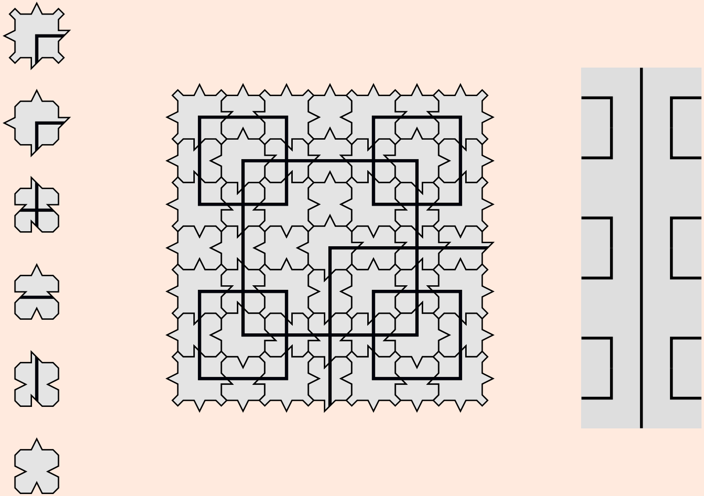

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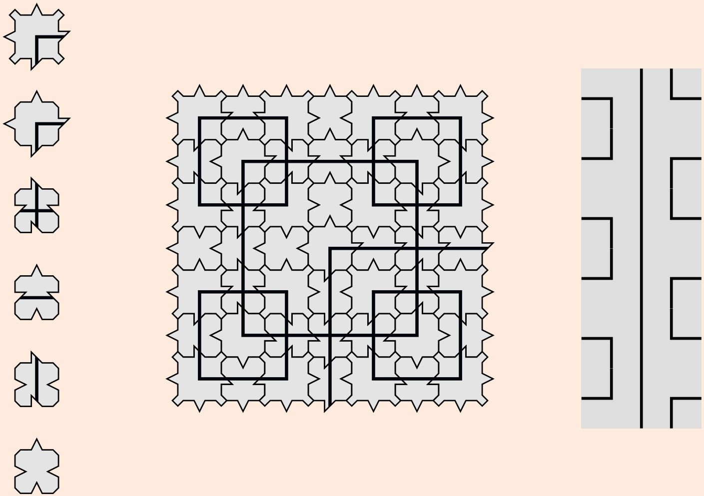

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Strenghtening the Structure

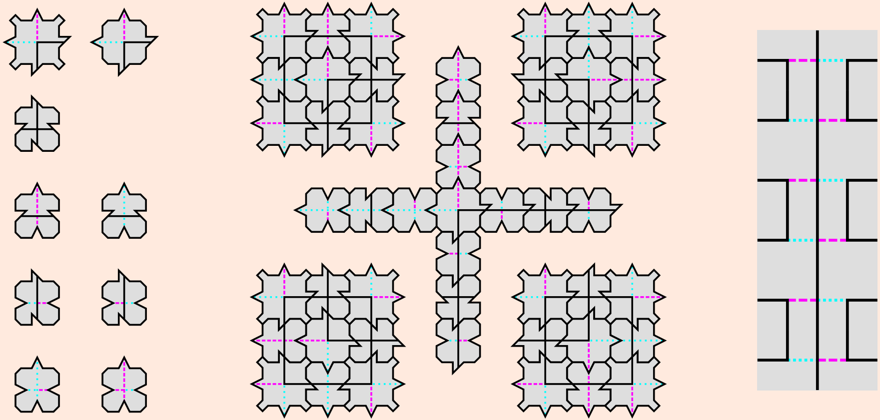


Figure 11: A Robinson variant, with strengthened local rules.

High-Density Quasi-Periodic Structure

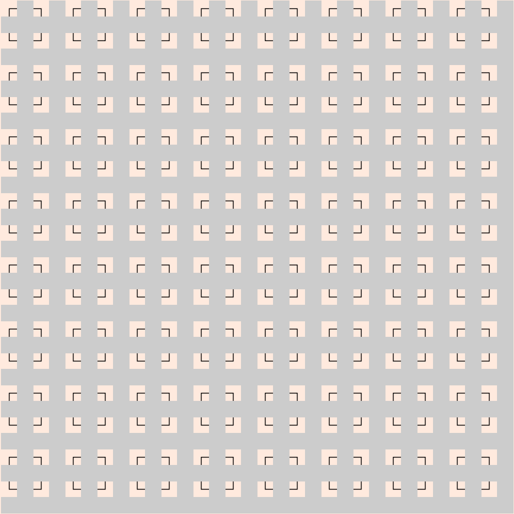
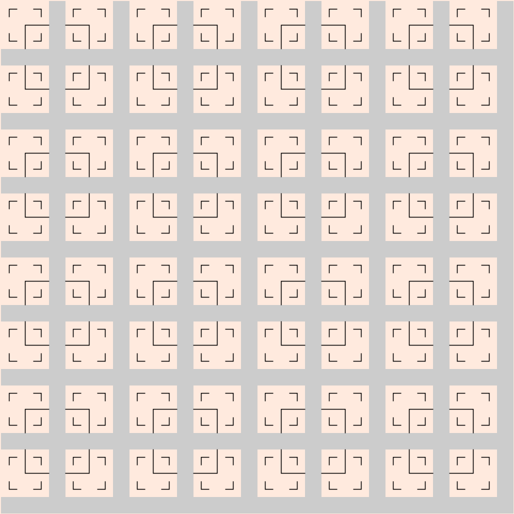


Figure 12: The density of the grid around N-macro-tiles goes to 0 as $N \to \infty$.

High-Density Quasi-Periodic Structure



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High-Density Quasi-Periodic Structure

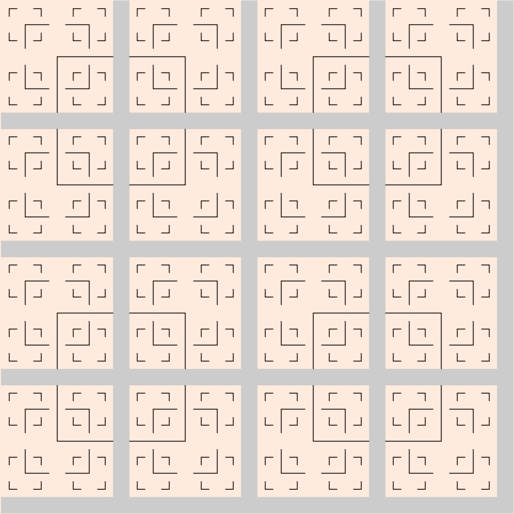

Figure 12: The density of the grid around N-macro-tiles goes to 0 as $N \to \infty$.

Stability for Aperiodic Tilings

Aperiodic Stability

Reconstruction Function for the Enhanced Tiling

Proposition [Gayral and Sablik, 2021, Proposition 7.7]

For any scale $N \geq 2$, the constant $C_N = 2^N - 1$ is such that for any integer n and any clear locally admissible pattern w on B_{n+C_N}, $w|_{B_n}$ is almost globally admissible, in the sense that up to a low-density grid, $w|_{B_n}$ is made of well-aligned and well-oriented N-macro-tiles.

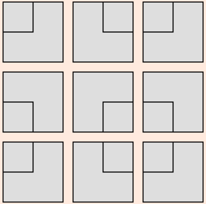

Figure 13: Family of well-aligned and well-oriented tiles.

Non-linear Polynomial Stability

Theorem [Gayral and Sablik, 2021, Proposition 7.8 and Theorem 7.9]

For any $\varepsilon > 0$, any scale N, and any measure $\mu = \pi_1^*(\lambda)$ with $\lambda \in \widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon)$:

$$d_B\left(\mu, \mathcal{M}_{\mathcal{F}}\right) \leq 96\left(2^{N+2}+1\right)^2 \varepsilon + \frac{1}{2^{N-1}}.$$

Hence, the SFT is f-stable with $f(\varepsilon) = 48\sqrt[3]{6\varepsilon}$.

Could we obtain faster bounds for an aperiodic tiling?

Stability for Aperiodic Tilings

Aperiodic Unstability

A Two-Coloured Robinson Tiling

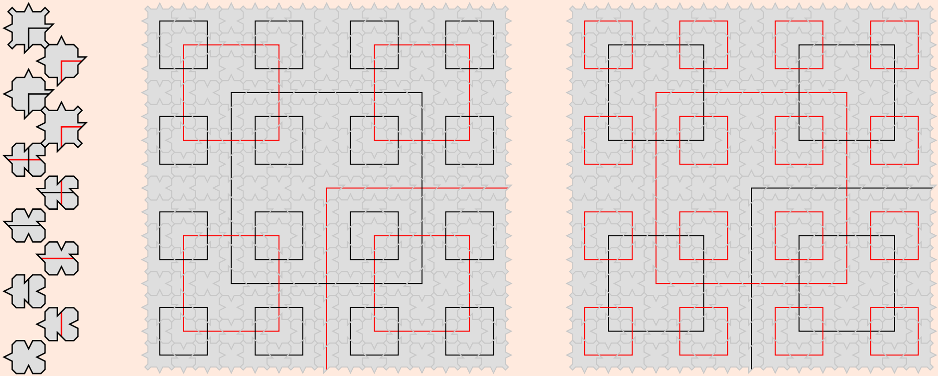

Figure 14: Two-coloured Robinson structure.

Unstable Colour Flips

Proposition [Gayral, 2021, Proposition 1]

The SFT Ω_{RB} is unstable.

More precisely, for any $\varepsilon > 0$, we have $\mu \in \mathcal{M}^{\mathcal{B}}_{RB}(\varepsilon)$ such that $d_B(\mu, \mathcal{M}_{RB}) \geq \frac{1}{8}$.

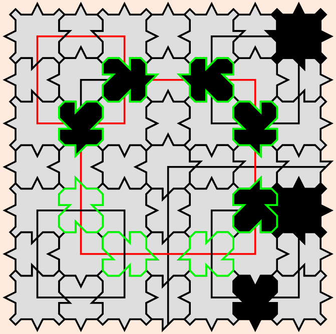

Figure 15: The Red-Black alternating structure allows for colour flips.

Undecidability of the Stability

Crash Course on Decidability Classes

Turing Machines and Decidability

Turing machines are a formal model equivalent to real-life computers and algorithms.

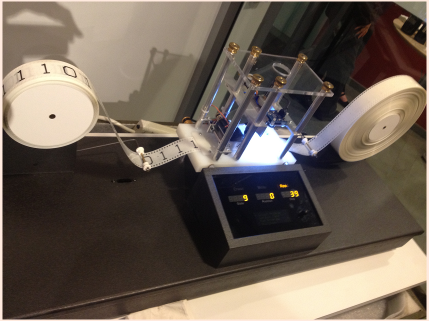

Figure 16: Real-life implementation of a Turing machine (Source: wikimedia.org)

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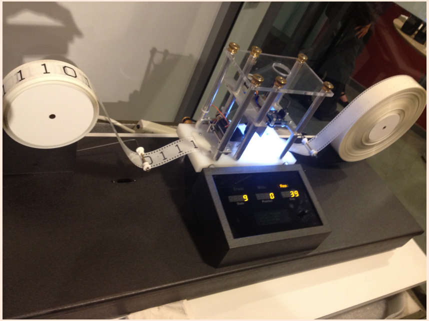

Figure 16: Real-life implementation of a Turing machine (Source: wikimedia.org)

A *decision problem* is a *yes/no* question.

A problem is *decidable* if there is an algorithm that answers it in finite time.

The Arithmetical Hierarchy of Undecidable Problems

- The halting problem P_{halt} on the algorithm φ is Σ_1-complete:

$$P(\varphi) \equiv \exists t \in \mathbb{N}, \varphi(0) \text{ terminates in } t \text{ steps or less}$$

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The Arithmetical Hierarchy of Undecidable Problems

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Definition (Class of Problems Π_k)

Consider a decision problem $P : \mathbb{N} \to \{0,1\}$.

We have $P \in \Pi_k$ if there is a decidable $\varphi : \mathbb{N}^{k+1} \to \{0,1\}$ such that:

$$P(x) \equiv \underbrace{\forall y_1 \in \mathbb{N}, \exists y_2 \in \mathbb{N}, \forall y_3 \in \mathbb{N}, \ldots,}_{k \text{ alternating quantifiers}} \varphi(x, y_1, \ldots, y_k)$$

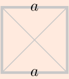
Undecidability of the Stability

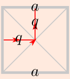
Σ_1-Hardness of the Problem

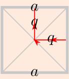
Turing Machine Space-Time Diagrams as Tilings

Consider a Turing machine $(Q, \Gamma, I, F, \delta)$ and define the following Wang tiles:

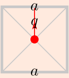
- For any letter $a \in \Gamma$ and any state $q \in Q$:



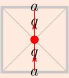




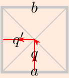
- For any letter $a \in \Gamma$ and initial state $q \in I$:



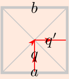
- For any letter $a \in \Gamma$ and final state $q \in F$:



- For any transition $\delta(a, q) = (b, q', L)$:



- For any transition $\delta(a, q) = (b, q', R)$:



Embedding Space-Time Diagrams into Robinson Tilings

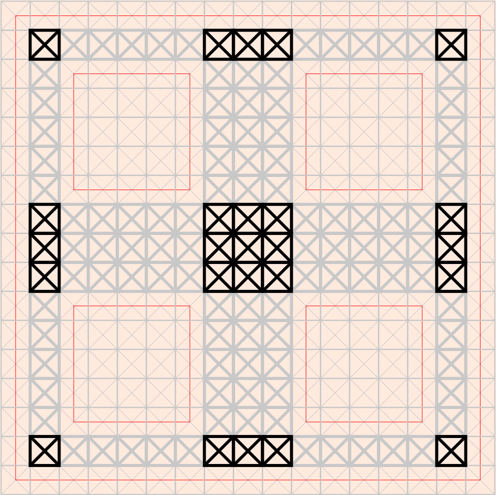


Figure 17: The free black tiles encode the diagram, the grey ones are communication channels.

A Four-Coloured Enhanced Robinson Tiling

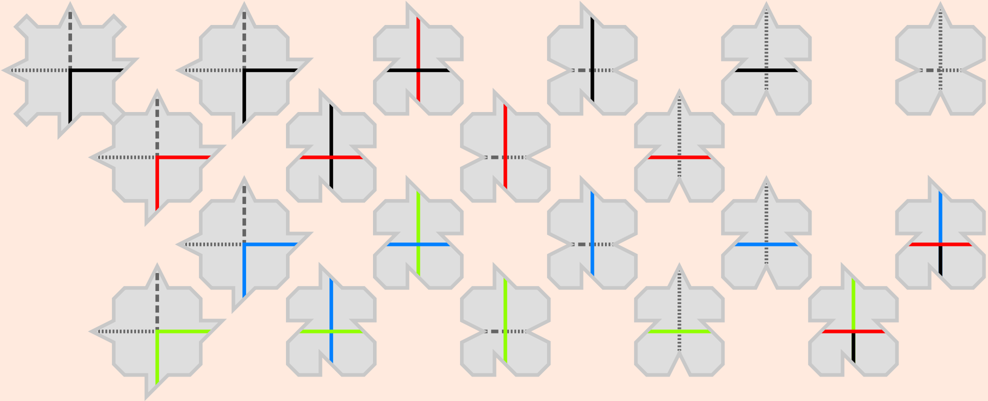


Figure 18: The tiling uses an enhanced Robinson structure.
It starts with Black bumpy tiles, alternates between Red and Black,
then may transition to an unstable Blue-Green regime.

Transition from the Red-Black to the Blue-Green Phase

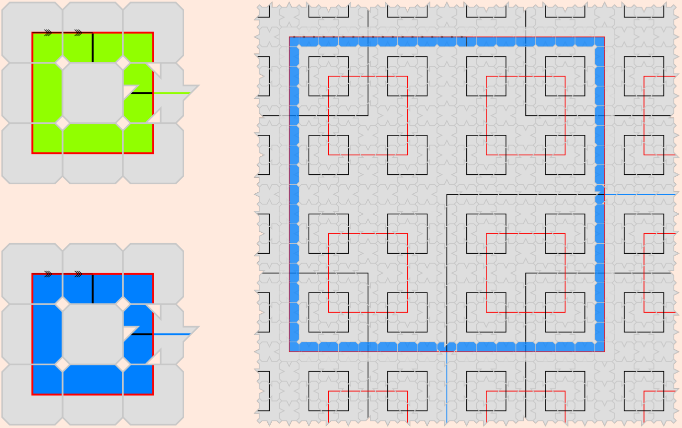


Figure 19: The transition appears *iff* there is a final state, and the colour choice propagates on the border.

Undecidability of the Stability

Theorem [Gayral, 2021, Theorem 1]

Denote $\mathcal{F}_T$ the SFT that embeds the Turing machine T into a Robinson tiling.

Then $\mathcal{F}_T$ is stable (for d_B on the class $\mathcal{B}$) iff T does not halt on the empty input.

In the stable case, $\mathcal{F}_T$ is polynomially stable.

Because the halting problem is Σ_1-hard, so is the question of unstability.

Theorem

Stability is Π_1-hard.

Idea for the Stable Case: All N-Macro-Tiles Are Mostly the Same

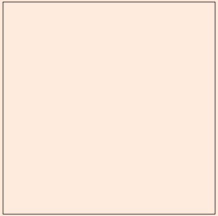


Figure 20: In a $2N$-macro-tile, only $O\left(12^N\right)$ tiles out of 16^N are ignored.

This gives a $16^N \times C\varepsilon + \left(\frac{3}{4}\right)^N$ bound on d_B at scale $2N$.

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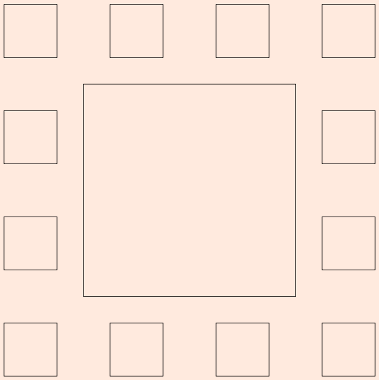

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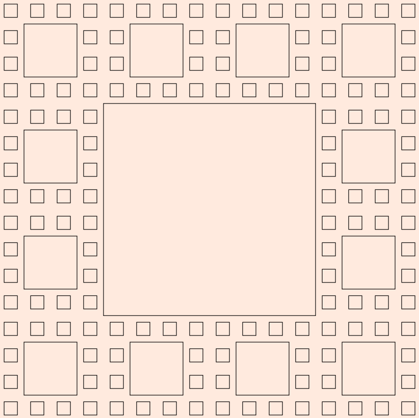

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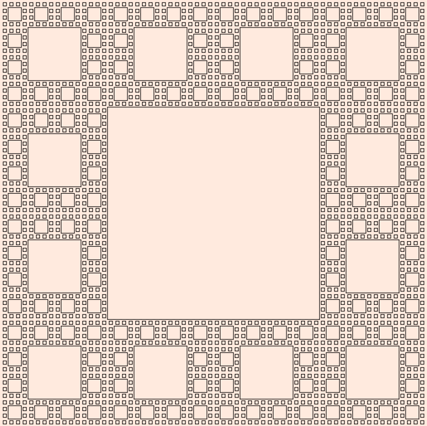

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Idea for the Unstable Case: Two Kinds of Widely Different N-Macro-Tiles

We can do the same Blue-Green colour flip as in our Red-Black unstable example.

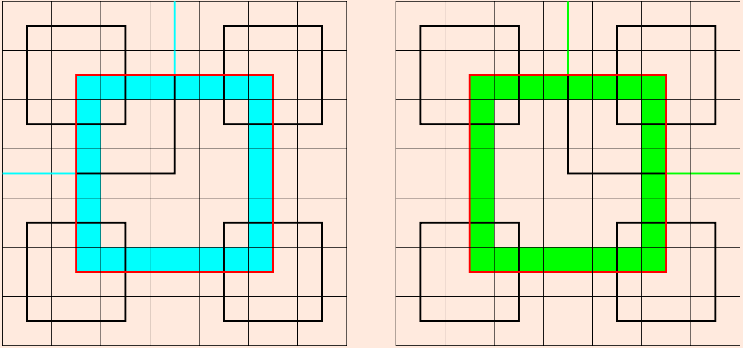

Figure 21: The transition scale plays the role of 1-macro-tiles for the Blue-Green phase.

If the Turing machine stops in the $2N$-macro-tiles,
we guarantee a $\Omega\left(\frac{1}{16^N}\right)$ density of differences between Blue and Green.

Dual Construction for Σ_1-Hard Stability

We now use the following Robinson structure:

- Encode two bits (a, b) in the central arm.
- The Red-Black bit a starts as Black and then alternates, for the Turing structure.
- The Blue-Green bit b starts freely and then alternates.
- Whenever the machine stops (necessarily a is Black), b *must* be Blue.

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Thence:

T halts	$\Rightarrow$	One kind of macro-tiles only at big-enough scales	$\Rightarrow$	Stable
T doesn't stop	$\Rightarrow$	Two kinds of different macro-tiles at all the scales	$\Rightarrow$	Unstable

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Theorem

Stability is Σ_1-hard.

Undecidability of the Stability

Climbing the Arithmetical Hierarchy

Toeplitz Encoding of a Sequence

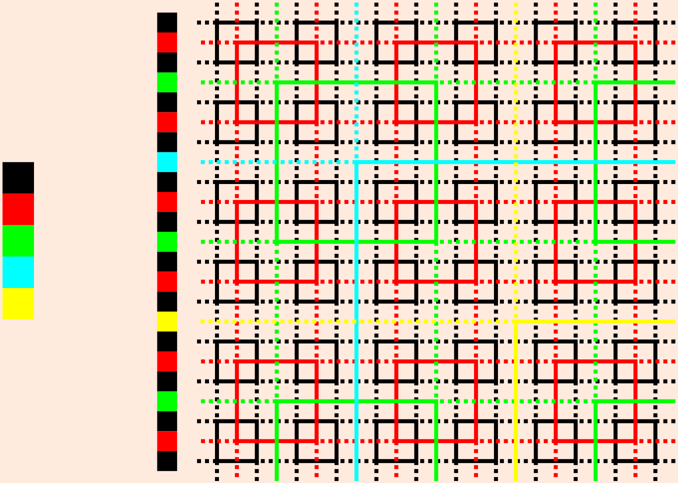

Figure 22: Toeplitz encoding of the sequence of colours on the left into the Robinson hierarchy.

Toeplitz Encoding of a Sequence

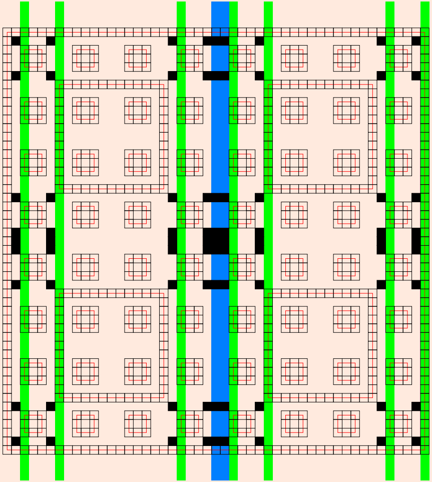

Figure 23: In practice, we see a finite prefix of the Toeplitz encoding as a read-only input.

Generalising the Σ_1-Hard Construction on Toeplitz Inputs

Consider a Turing machine T with the alphabet $\Gamma = \Sigma \sqcup \{\#\}$.

We encode a word $w \in \Sigma^*\{\#\}^*$ of length N in a $(2N+1)$-macro-tile.

We have three phases in the Robinson hierarchy:

1. Decoding of the Toeplitz sequence into a word $w \in \Sigma^*$.

$\Downarrow$

Ignition of the unstable Blue-Green bit.

$\Downarrow$

2. Partial computation of T on the input w.

$\Downarrow$

Freezing of the now stable Blue-Green bit.

$\Downarrow$

3. T halts on w.

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Freezing of the now stable Blue-Green bit.

$\Downarrow$

3. T halts on w.

Stability is Π_2-Hard

We have 3 situations globally speaking:

- Infinite decoding of an input $w \in \Sigma^{\mathbb{N}}$: stable.
- Halting of T on $w \in \Sigma^*$: stable.
- Infinite computation of T on $w \in \Sigma^*$: unstable through colour flips.

We have a family of bounds of the form:

$$d_B(\mu_\varepsilon, \mathcal{M}_{\mathcal{F}_T}) \leq 16^{\varphi(N)} \times C\varepsilon + \left(\frac{3}{4}\right)^N,$$

with $\varphi(N)$ the scale at which T has halted on all the inputs of length at most N.

If T halts on all the inputs, the bound still goes to 0 as $\varepsilon \to 0$,
but cannot be explicit as φ can be bigger than any computable function.

Theorem

$\mathcal{F}_T$ is stable iff T stops on all its entries, which is a Π_2-complete problem.

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Stability is Π_2-Hard

We have 3 situations globally speaking:

- Infinite decoding of an input $w \in \Sigma^{\mathbb{N}}$: stable.
- Halting of T on $w \in \Sigma^*$: stable.
- Infinite computation of T on $w \in \Sigma^*$: unstable through colour flips.

We have a family of bounds of the form:

$$d_B\left(\mu_\varepsilon, \mathcal{M}_{\mathcal{F}_T}\right) \leq 16^{\varphi(N)} \times C\varepsilon + \left(\frac{3}{4}\right)^N,$$

with $\varphi(N)$ the scale at which T has halted on all the inputs of length at most N.

If T halts on all the inputs, the bound still goes to 0 as $\varepsilon \to 0$,
but cannot be explicit as φ can be bigger than any computable function.

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Undecidability of the Stability

Finding an Upper Bound

Could the Problem be Π_2-Complete?

Stability of $\Omega_{\mathcal{F}}$: $\forall \delta > 0, \exists \varepsilon > 0, \sup_{\mu \in \mathcal{M}_{\mathcal{F}}^{\mathcal{B}}(\varepsilon)} d_B(\mu, \mathcal{M}_{\mathcal{F}}) \leq \delta$.

By monotonicity we can consider $\varepsilon, \delta \in \mathbb{Q}^{+*}$.

Theorem

The SFT $\Omega_{\mathcal{F}}$ is stable iff it satisfies the following formula:

$$\forall \delta \in \mathbb{Q}^{+*}, \exists \varepsilon \in \mathbb{Q}^{+*}, \forall \rho \in \mathbb{Q}^{+*}, \exists \gamma \in \mathbb{Q}^{+*}, \gamma \leq \rho,$$
$$\forall (w,b) \in \widetilde{\mathcal{W}_{\mathcal{F}}^{\varepsilon}}(\gamma), \exists w_0 \in \mathcal{W}_{\mathcal{F}}(\rho), \exists (w_1, w_2) \in (\mathcal{A}^2)^{U_{\psi(\rho,|\mathcal{A}^2|,d)}},$$
$$\left[d^+_{|\mathcal{A}|}\left(\widehat{\delta_{w_1}}, \widehat{\delta_w}\right) < 3\rho\right] \wedge \left[d^+_{|\mathcal{A}|}\left(\widehat{\delta_{w_2}}, \widehat{\delta_{w_0}}\right) < 3\rho\right] \wedge \left[\widehat{\delta_{(w_1,w_2)}}(\Delta) \leq \delta + |\mathcal{A}|^2\rho\right],$$

with $\widetilde{\mathcal{W}_{\mathcal{F}}^{\varepsilon}}(\gamma)$ and $\mathcal{W}_{\mathcal{F}}(\rho)$ being finite computable sets, and $U_{\psi(\rho,|\mathcal{A}^2|,d)}$ a computable function.

Hence a Π_4 upper bound on the problem.

Bibliography

Durand, B., Romashchenko, A., and Shen, A. (2012).
Fixed-point tile sets and their applications.
Journal of Computer and System Sciences, 78(3):731–764.

Gayral, L. (2021).
The Besicovitch-stability of noisy tilings is undecidable.
hal.archives-ouvertes.fr/hal-03233596.

Gayral, L. and Sablik, M. (2021).
On the Besicovitch-stability of noisy random tilings.
arxiv.org/abs/2104.09885v2.

THE END OF PRESENTATION

ONE MORE SLIDE:

Thank you.