

# Computational Complexity of Chaotic Gibbs Measures

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## Chaotic Gibbs Measures

- General Framework

- Chaos Ensues

## Bounding the Chaotic Complexity

## Controlling Markers Distribution

## Building an Appropriate Structure

- The Robinson Tiling(s)

- Structure for Entropy Control

# Chaotic Gibbs Measures

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General Framework

# Subshifts of Finite Type and Forbidden Patterns

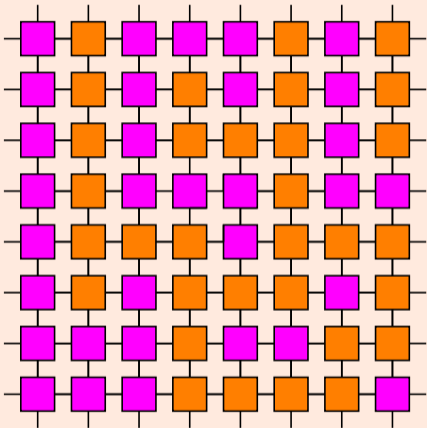
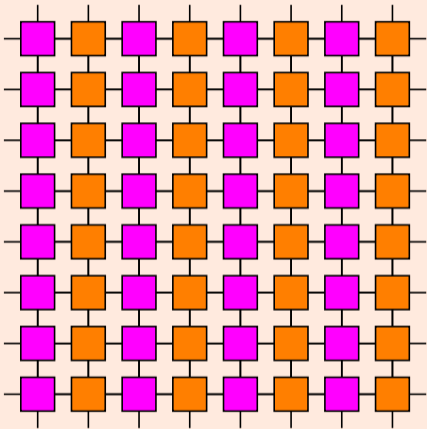


Figure 1: Example of configuration,

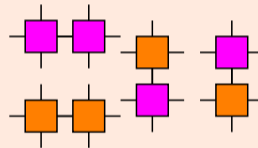
- Group  $G = \mathbb{Z}^2$  with 2 generators.
- Alphabet  $\mathcal{A} = \{\text{pink}, \text{orange}\}$ .

# Subshifts of Finite Type and Forbidden Patterns

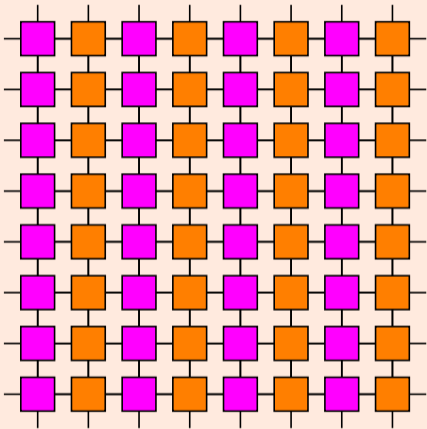


**Figure 1:** Example of configuration, without occurrences of the forbidden patterns.

- Group  $G = \mathbb{Z}^2$  with 2 generators.
- Alphabet  $\mathcal{A} = \{\text{pink}, \text{orange}\}$ .
- Finite set of forbidden patterns  $\mathcal{F}$ :

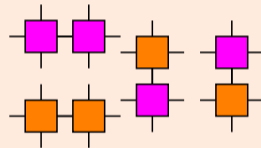


# Subshifts of Finite Type and Forbidden Patterns



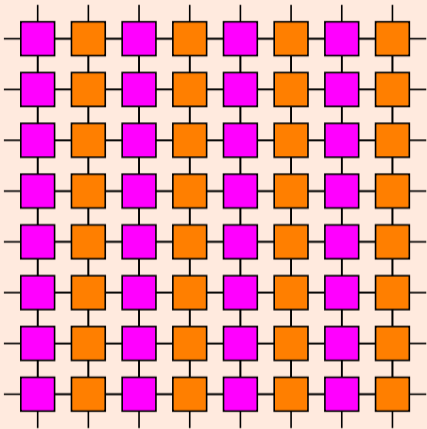
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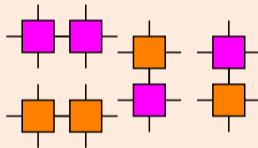
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# Subshifts of Finite Type and Forbidden Patterns



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- The SFT is the space  $\Omega_{\mathcal{F}} \subset \mathcal{A}^G$  of such configurations.
- Denote  $\mathcal{M}_{\mathcal{F}}$  the space of translational-invariant measures.

# Gibbs Measures

Local viewpoint on a finite phase space  $\Omega$ :

- Energy  $E : \Omega \rightarrow \mathbb{R}^+$ ,
- Inverse Temperature  $\beta \in \mathbb{R}^+$ ,
- Gibbs Measure  $\mu_\beta(\omega) := \frac{1}{Z} \exp(-\beta E(\omega))$ .



# Gibbs Measures

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Global translational-invariant viewpoint on  $\Omega_{\mathcal{A}} = \mathcal{A}^{\mathbb{Z}^2}$ :

- Finite range potential  $f : \mathcal{A}^{I_r} \rightarrow \mathbb{R}^+$  with  $I_r = \llbracket -r, r \rrbracket^2$ ,
- Pressure function  $P(\mu, \beta) = h(\mu) - \beta \mu(f)$  for  $\mu \in \mathcal{M}_{\mathcal{A}}$ ,
- Gibbs measures  $\mathcal{G}(\beta) = \operatorname{argmax}_{\mu} P(\mu, \beta)$ ,
- Ground states  $\mathcal{G}(\infty) := \operatorname{Acc}(\mathcal{G}(\beta), \beta \rightarrow \infty)$ .

# A Link Between Worlds

$\mathcal{F}$  (forbidden patterns)  $\rightarrow$   $\Omega_{\mathcal{F}}$  (tilings)  $\rightarrow$   $\mathcal{M}_{\mathcal{F}}$  (measures)

$f$  (potential)  $\rightarrow$   $\mathcal{G}(\beta)$  (Gibbs measures)  $\rightarrow$   $\mathcal{G}(\infty)$  (ground states)

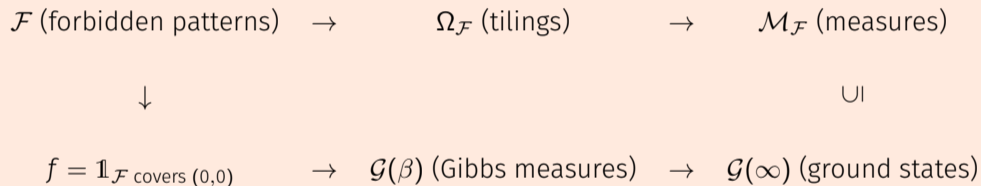
# A Link Between Worlds

$\mathcal{F}$  (forbidden patterns)  $\rightarrow$   $\Omega_{\mathcal{F}}$  (tilings)  $\rightarrow$   $\mathcal{M}_{\mathcal{F}}$  (measures)

$\downarrow$

$f = \mathbf{1}_{\mathcal{F} \text{ covers } (0,0)}$   $\rightarrow$   $\mathcal{G}(\beta)$  (Gibbs measures)  $\rightarrow$   $\mathcal{G}(\infty)$  (ground states)

# A Link Between Worlds



# Chaotic Gibbs Measures

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Chaos Ensues

# Weak Chaoticity

## Definition

The model is said to be *weakly chaotic* if:

$$\text{For any choice of } \mu_\beta \in \mathcal{G}(\beta), \# \text{Acc}(\mu_\beta, \beta \rightarrow \infty) \geq 2.$$

**Theorem [Chazottes and Shinoda, 2020, Barbieri et al., 2022]**

There is a 2D set of forbidden patterns  $\mathcal{F}$  that induces a weakly chaotic system.

# Strong Chaoticity

## Definition

Assume  $\#\mathcal{G}(\infty) \geq 2$ . The model is said to be *strongly chaotic* if:

For any choice of  $\mu_\beta \in \mathcal{G}(\beta)$ ,  $\text{Acc}(\mu_\beta, \beta \rightarrow \infty) = \mathcal{G}(\infty)$ .

## Theorem [Gayral, Sablik and Taati, 2022(?)]

There is a 2D set of forbidden patterns  $\mathcal{F}$  that induces a strongly chaotic system.

## “Conjecture” / WIP

What's more, we can obtain  $\mathcal{G}(\infty) = f(X)$ ,

with  $X$  being any  $\Pi_2$ -computable connected subset of  $\mathcal{M}(\{0, 1\}^{\mathbb{N}})$ ,

and  $f : \mathcal{M}(\{0, 1\}^{\mathbb{N}}) \rightarrow \mathcal{M}_{\mathcal{F}}$  an appropriate convex bijection, an affine homeomorphism.

# Bounding the Chaotic Complexity

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# Computational Complexity of Uncountable Sets

Let  $(X, d)$  a metric space with a countable dense basis  $\mathcal{B}$ .

## Definition

Let  $Y \subset X$  be a closed set and  $\mathcal{N}(Y) := \{(x, r) \in \mathcal{B} \times \mathbb{Q}^{+*}, \overline{B(x, r)} \cap Y \neq \emptyset\}$ .

The set  $Y$  is said to be  $\Pi_k$ -computable *iff* the countable set  $\mathcal{N}(Y)$  is, *i.e.* there is a computable  $\varphi$  such that:

$$(x, r) \in \mathcal{N}(Y) \Leftrightarrow \forall y_1, \exists y_2, \forall y_3, \dots, \varphi(x, r, y_1, \dots, y_k)$$

Here, for  $\sigma$ -invariant measures  $\mathcal{M}_{\mathcal{A}}$  with the weak-\* topology, we use the periodic measures  $\widehat{\delta}_w$ , with  $w \in \mathcal{A}^{\llbracket 0, n-1 \rrbracket^d}$ , as a basis  $\mathcal{B}$ .

# Equivalent Characterisation of $\Pi_2$

## Theorem [Hellouin de Menibus and Sablik, 2018]

*There is a characterisation of  $\Pi_2$ -computable sets through accumulation points:*

$Y \in \Pi_2$   $\Leftrightarrow$   $Y = \text{Acc}(x_n, n \rightarrow \infty)$  with  $(x_n) \in \mathcal{B}^{\mathbb{N}}$  computable.

$Y \in \Pi_2$  and connected  $\Leftrightarrow$   $Y = \text{Acc}([x_n, x_{n+1}], n \rightarrow \infty)$  with  $(x_n) \in \mathcal{B}^{\mathbb{N}}$  computable.

# Computable Approximations of Gibbs Measures

Let  $f$  be a finite range potential, and  $P(\mu, \beta) = h(\mu) - \beta\mu(f)$  the corresponding pressure.

## Lemma

The maximal pressure function  $\beta \mapsto \max_{\mu} P(\mu, \beta)$  is computable.

There is a computable family of finite sets  $(G_{a,b}^n)_{n \in \mathbb{N}, a \leq b \in \mathbb{Q}^{+*}}$  such that:

- $G_{a,b}^n$  is a set of dyadic measures on  $\llbracket 0, n-1 \rrbracket^d$  with near-maximal pressure (repeated independently in all directions and averaged),
- $\bigcup_{a \leq \beta \leq b} \mathcal{G}(\beta) = \text{Acc}(G_{a,b}^n, n \rightarrow \infty)$ .

# General Upper Bound

## Theorem

We have  $\overline{B(x, r)} \cap \mathcal{G}(\infty) \neq \emptyset$  iff:

$$\forall \varepsilon \in \mathbb{Q}^{+*}, \forall a \in \mathbb{N}, \quad \exists b \in \mathbb{N}_{>a}, \exists n_0 \in \mathbb{N}, \quad \forall n \in \mathbb{N}_{>n_0}, \\ \exists y \in G_{b-1, b+1}^n, \overline{B(y, \varepsilon)} \cap \overline{B(x, r)} \neq \emptyset.$$

Consequently, we have a  $\Pi_3$  upper bound on the complexity of  $\mathcal{G}(\infty)$ .

# Strongly Chaotic Upper Bound

Assume now that the system is strongly chaotic.

## Lemma

*There is a sequence of  $\beta_k \rightarrow \infty$  such that  $\text{diam}(\mathcal{G}(\beta_k)) \rightarrow 0$ .*

*Without loss of generality, we have rational parameters  $\beta_k \in \mathbb{Q}$ .*

## Theorem

*We have  $\overline{B(x, r)} \cap \mathcal{G}(\infty) \neq \emptyset$  iff:*

$$\forall \varepsilon \in \mathbb{Q}^{+*}, \forall \beta_0 \in \mathbb{Q}^{+*}, \quad \exists \beta \in \mathbb{Q}_{>\beta_0}^{+*}, \exists y \in \mathcal{B}, \\ \mathcal{G}(\beta) \subset B(y, \varepsilon) \text{ and } B(y, \varepsilon) \cap \overline{B(x, r)} \neq \emptyset.$$

Consequently, we have a  $\Pi_2$  upper bound on the complexity of  $\mathcal{G}(\infty)$ .

# Perspectives

The difference in the general and the chaotic bound suggests a link between:

- the dynamic notion of chaotic systems of Gibbs measures,
- the computational complexity of the zero-temperature limit set itself.

Ultimately, we would like to prove the optimality of both bounds.

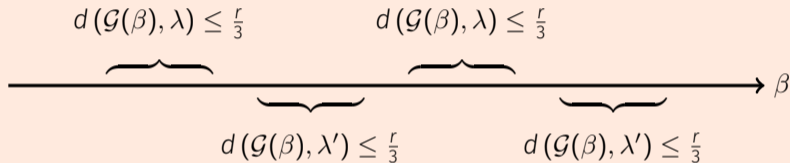
A first step in this direction is to implement connected  $\Pi_2$  sets as strongly chaotic limits.

# Controlling Markers Distribution

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# General Idea of the Weak Chaoticity

We have two measures  $\lambda \neq \lambda'$  s.t.  $d(\lambda, \lambda') \geq r$  and:

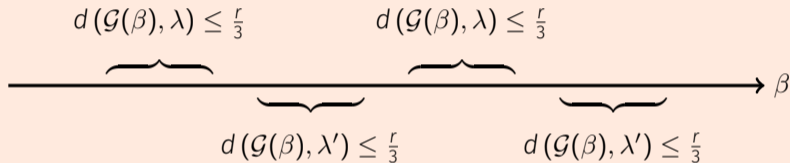


**Figure 2:** Alternating between incompatible behaviours on non-overlapping intervals.



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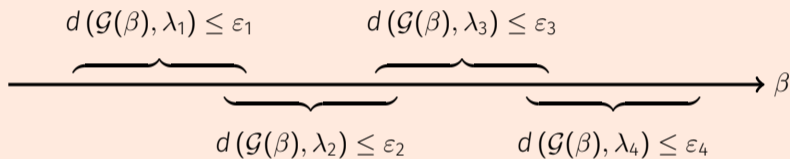


**Figure 2:** Alternating between incompatible behaviours on non-overlapping intervals.

Thus  $\text{Acc}(\mu_\beta)$  must intersect the disjoint neighbourhoods of both  $\lambda$  and  $\lambda'$ .

# Moving Onto Strong Chaoticity

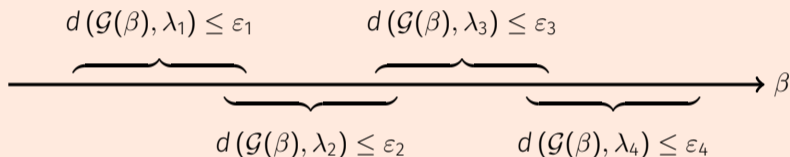
We want  $(\lambda_n)$  and  $\varepsilon_n \rightarrow 0$  s.t.:



**Figure 3:** Phasing through similar behaviours on overlapping intervals.

# Moving Onto Strong Chaoticity

We want  $(\lambda_n)$  and  $\varepsilon_n \rightarrow 0$  s.t.:



**Figure 3:** Phasing through similar behaviours on overlapping intervals.

Thus  $\text{Acc}(\mu_\beta) = \mathcal{G}(\infty) = \text{Acc}(\lambda_n, n \rightarrow \infty)$ .

# Control of Markers on a Temperature Interval

## Definition

We consider a *marker* set  $Q \subset \mathcal{A}^B$  (with  $B = l_k$ ), made of non-overlapping patterns, such that any locally admissible tiling  $\omega \in \mathcal{A}^{l_{(2+\rho)k}}$  must contain a marker somewhere.

## Theorem [Adapted from Chazottes-Hochman]

Denote  $G_n$  the admissible tilings of  $I_n$ , and  $\mu_Q$  the cond. measure on  $Q$  of  $\mu|_{\mathcal{A}^B}$ . We have constants  $C, C'$  s.t. for any marker set  $Q$  and  $\varepsilon, \kappa > 0$ , if:

$$\frac{\ln(\#G_n)}{\#I_n} \geq (1 - \kappa) \frac{\ln(\#Q)}{\#B} \quad \text{and} \quad C \frac{\#B}{\varepsilon} \leq \beta \leq C'n\varepsilon,$$

then for any  $\mu \in \mathcal{G}(\beta)$ :

$$\mu(Q \text{ covers } 0) = 1 - O(\varepsilon + \rho) \quad \text{and} \quad H(\mu_Q) \geq (1 - 2\kappa) \ln(\#Q) - H(\kappa) - O(\varepsilon + \rho).$$

# Building an Appropriate Structure (*aka* LEGO for Grownups)

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The Robinson Tiling(s)

# Folkloric Robinson Tiling (Non-Overlapping Markers)



**Figure 4:** Hierarchical structure of the Robinson tiling.

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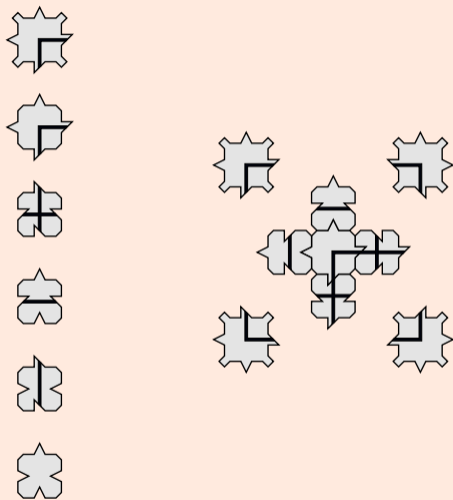


Figure 4: Hierarchical structure of the Robinson tiling.



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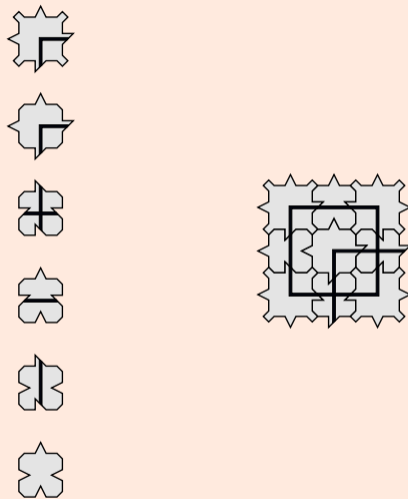
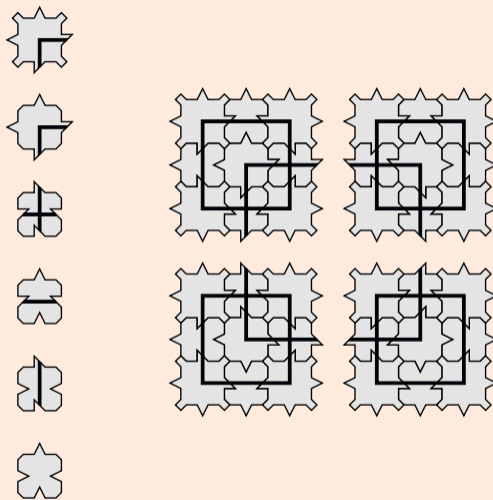


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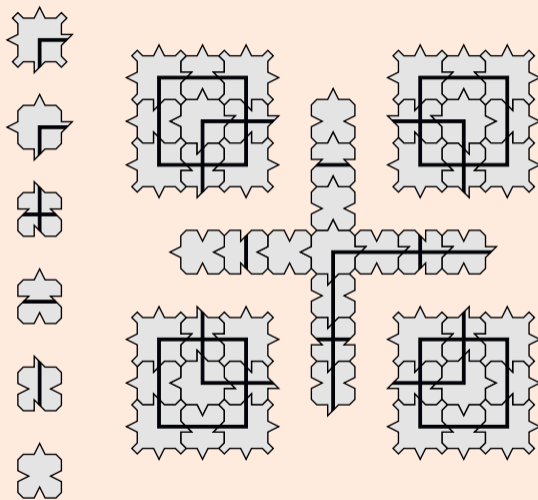


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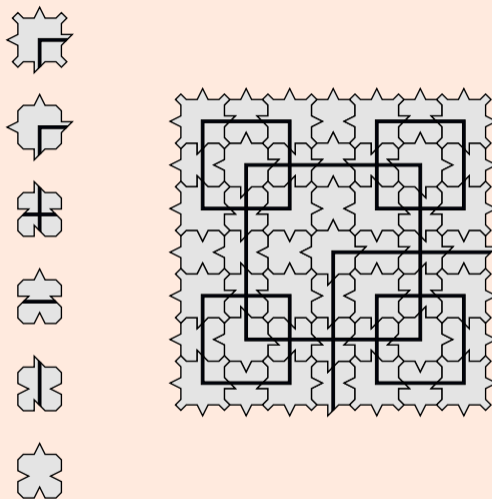


Figure 4: Hierarchical structure of the Robinson tiling.

# Enhanced Robinson Tiling (Markers with Reconstruction)

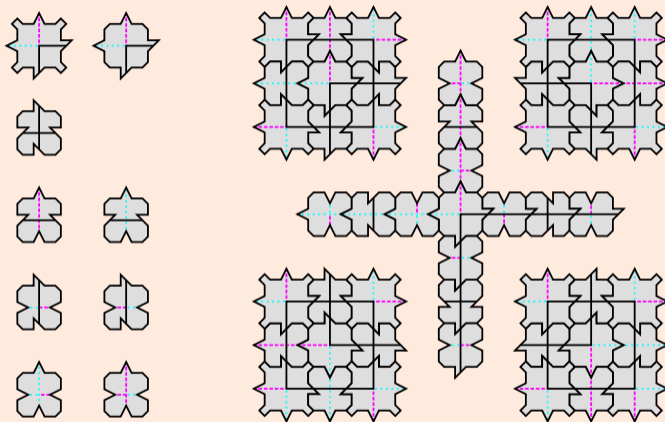
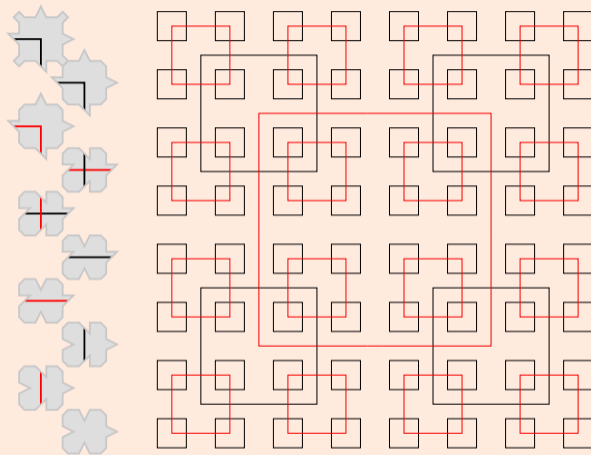


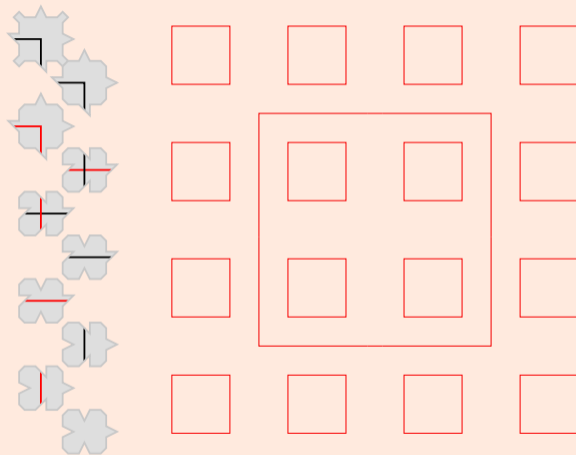
Figure 5: A Robinson variant, with strengthened local rules.

# Two-Coloured Robinson for Turing Machines (Markers with Computation Area)



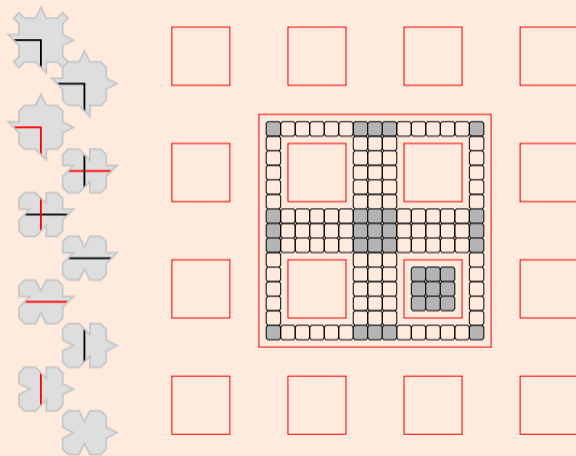
**Figure 6:** Alternating Red-Black structure,

# Two-Coloured Robinson for Turing Machines (Markers with Computation Area)



**Figure 6:** Alternating Red-Black structure,

# Two-Coloured Robinson for Turing Machines (Markers with Computation Area)



**Figure 6:** Alternating Red-Black structure, with a sparse computation area.



# Structural Properties of the Base Layer

- The  $n$ -macro-tile has a length  $l_n = 2^n - 1$ .
- The  $n$ -macro-tiles are non-overlapping.
- Any locally admissible window of length  $2l_n + c$  contains a  $n$ -macro-tile.  
(adapted from [Gayral and Sablik, 2021, Proposition 7.7])
- The  $N$ -th Red square occurs in a  $(2N + 1)$ -macro-tile.
- The  $N$ -th Red square has a length  $4^N + 1$ .
- The  $N$ -th Red square has a sparse computing area of horizon  $2^N + 1$ .

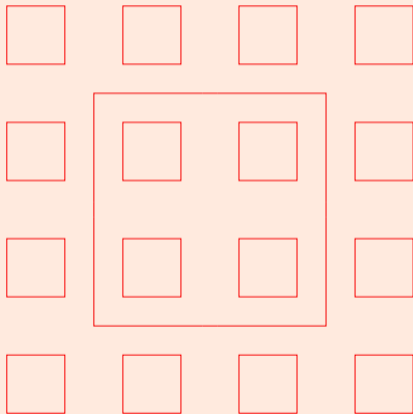
# Building an Appropriate Structure (*aka* LEGO for Grownups)

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Structure for Entropy Control

# Hot and Frozen Areas

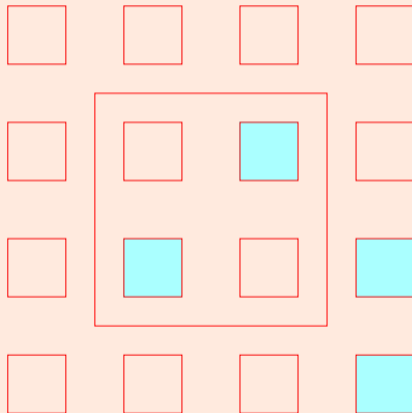
Red squares may be Blocking, with a Hot exterior and Frozen core.  
The rest must locally synchronise on Hot or Frozen.



**Figure 7:** Admissible configurations for Hot and Frozen squares.

# Hot and Frozen Areas

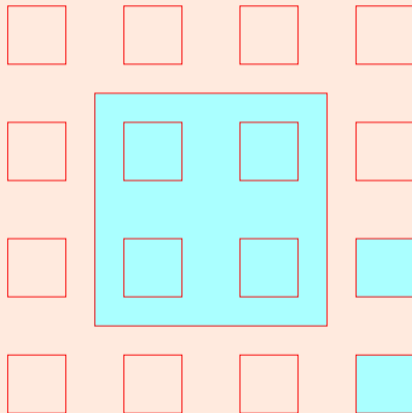
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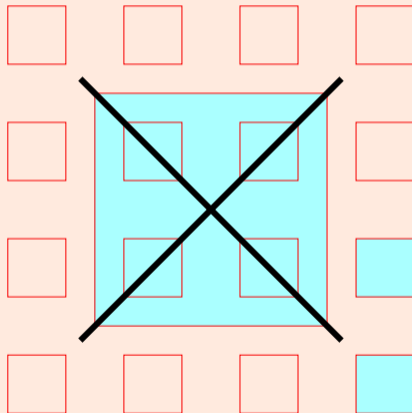
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**Figure 7:** Admissible configurations for Hot and Frozen squares.

# Blockable Scales

We (can) unary encode  $N$  as an input for computations in the  $N$ -th Red square.  
We check whether  $N = 3^k$ . If not, the Red square *cannot* be Blocking.



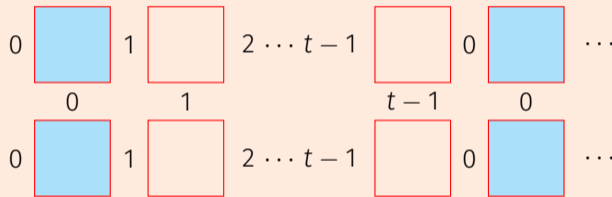
**Figure 8:** The 2nd scale of Red squares cannot be Blocking.

# Scales for the Marker Sets

- $Q_k$  the set of  $(2 \times 3^k + 1)$ -macro-tiles (Robinson layer) on the window  $B_k$ , the  $3^k$ -th scale of locally admissible tiles with Red squares.
- A  $(k + 1)$ -marker is a grid of  $16^{3^k} \times 16^{3^k}$  smaller  $k$ -markers.
- These gaps in scale will allow for a control on the entropy, on the speed of convergence of  $\frac{\ln(\#Q_k)}{\#B_k} \rightarrow 0$ .

# Odometer

We implement an odometer in  $k$ -markers, that cycles with period  $t_k = 2^{\lfloor \log_2(\ln(k)) \rfloor}$ , so that Red squares are Blocking once for each cycle.

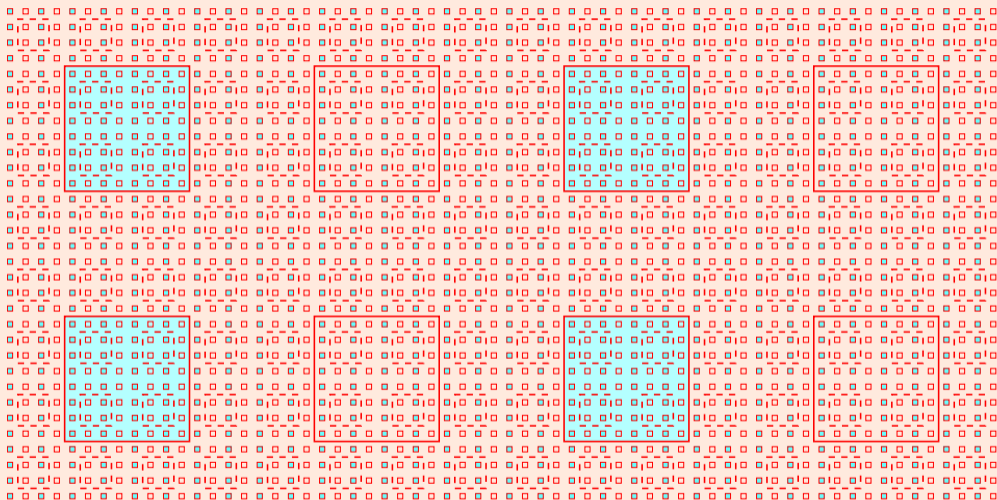


**Figure 9:** The repartition of Frozen squares is forced by the odometer.

The Red square of a  $(k + 1)$ -marker initialises  $k$ -markers at 0 on one side.



# Repartitionition of Frozen Tiles



**Figure 10:** Approximation of a *Total Perspective Vortex*.  
 (One 2-marker would be a  $4096 \times 4096$  grid of such 1-markers.)

# Repartition of Frozen Tiles

The average scale of Blocking squares in a  $k$ -marker goes to  $\infty$  as  $k \rightarrow \infty$ .

## Lemma

Fix a microscopic scale  $m$ .

The proportion of non-Frozen  $m$ -markers in a  $k$ -marker is of order:

$$\prod_{j=m+1}^k \left(1 - \frac{1}{4t_j}\right) \xrightarrow[k \rightarrow \infty]{} 0.$$

Thus, generically, a tiling  $\omega \in \Omega_{\mathcal{F}}$  is totally Frozen.

# Encoding Words

Encode a letter on Red lines so that:

- Blocking and Hot squares are labelled 0,
- Frozen squares are labelled  $\pm 1$ ,
- Neighbouring Frozen squares synchronise their bit.

Generically, a (Frozen) tiling  $\omega \in \Omega_{\mathcal{F}}$  encodes a sequence of bits in  $\{\pm 1\}^{\mathbb{N}}$ .

# Counting Markers

Let  $Q_k = Q_k^H \sqcup Q_k^B \sqcup Q_k^F$  depending on whether the Red square is Hot, Blocking or Frozen.

## Proposition

We have:

- $\#Q_k^H \approx C_k^{16^{3^k}}$  with  $2^{4^{-k}} \leq C_k \leq 2$ ,
- $\#Q_k^B \approx (\#Q_k^H)^{\frac{3}{4}}$ ,
- $\#Q_k^F \leq C^{4^{3^k}}$  for some  $C > 1$ .

Thus,  $\#Q_k \approx \#Q_k^H$ .

Using this result along with the bound on  $H(\mu_{Q_k})$ ,  
we conclude that  $\mu_{Q_k}$  is close to the uniform distribution on  $Q_k^H$ .






# Forcing a Distribution on Words?

Now that we have a well-behaved structure,  
we want to run a Turing machine in each Blocking square,  
to force a distribution on the encoded words.

This will easily give us strongly chaotic examples.

The next step will be to study carefully the kind of Turing machines we can use,  
to conclude on the kind of limit sets  $\mathcal{G}(\infty)$  we can have.

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# THE END OF PRESENTATION

ONE MORE SLIDE:

Thank you.