A Short Hike Through Symbolic Dynamics
And the Random Noise Therein

Léo Gayral
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Joint work with Mathieu Sablik
IMT, Université Toulouse III Paul Sabatier
Folklore on Symbolic Dynamics

Subshifts of Finiteness Type (SFTs)

(A)periodicity

From Configurations to Measures

Moving onto Noisy Tilings

Stability for Periodic Tilings

Stability for Aperiodic Tilings

The Robinson Tiling

Aperiodic Stability

Aperiodic Unstability

Closing Thoughts
Folklore on Symbolic Dynamics

Subshifts of Finity Type (SFTs)
Symbolic Dynamics on Groups

Dynamical System:
- Phase Space $\Omega$,
- Operators $(f_k : \Omega \to \Omega)$.

Symbolic Dynamics:
- Group $G$ with generators $(a_k)$,
- Finite Alphabet $A$,
- Phase Space $\Omega = A^G$,
- Shift Operators $\sigma_k$,
  s.t. $\sigma_k(\omega)_g = \omega - a_k + g$.  

Moving onto Noisy Tilings
Stability for Periodic Tilings
Stability for Aperiodic Tilings
Closing Thoughts
Subshifts of Finite Type and Forbidden Patterns

- Group $G = \mathbb{Z}^2$ with 2 generators.
- Alphabet $\mathcal{A} = \{\text{□}, \text{□}\}$.

Figure 1: Example of configuration,
Subshifts of Finite Type and Forbidden Patterns

• Group $G = \mathbb{Z}^2$ with 2 generators.

• Alphabet $\mathcal{A} = \{\text{pink square}, \text{orange square}\}$.

• Finite set of forbidden patterns $\mathcal{F}$:

Figure 1: Example of configuration, without occurrences of the forbidden patterns.
Subshifts of Finite Type and Forbidden Patterns

- Group $G = \mathbb{Z}^2$ with 2 generators.
- Alphabet $\mathcal{A} = \{\text{□}, \text{□}\}$.
- Finite set of forbidden patterns $\mathcal{F}$:

  ![Forbidden Patterns](image)

- The SFT is the space $\Omega_{\mathcal{F}} \subset A^G$ of such configurations.

**Figure 1:** Example of configuration, without occurrences of the forbidden patterns.
Why SFTs?

- Continuous Dynamical Systems:
  - Represent orbits through a finite/countable amount of information.

- Statistical Physics:
  - Ising model,
  - Domino models,
  - Quasicrystal models.

- Information Theory:
  - Space-time diagrams of cellular automata,
  - Computing model encoding Turing machines with a geometrical structure.
Folklore on Symbolic Dynamics

(A)periodicity
Figure 2: A periodic configuration,
Figure 2: A periodic configuration, characterised by a base hypercube that repeats in all directions.
Figure 3: Hierarchical structure of the Robinson tiling.
Aperiodic SFT

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**Figure 4:** This tileset forces no specific behaviour on admissible configurations.
Folklore on Symbolic Dynamics

From Configurations to Measures
Denote $\mathcal{M}_F$ the $\sigma$-invariant measures on $\Omega_F$, such that $\sigma_k^*(\mu) = \mu$ for any $k$.

Usual Ergodic Theory results, such as Birkhoff’s pointwise convergence theorem, naturally extend to this $d$-dimensional setting.
What About Noise?

There are several ways of adding noise to tilings.

- Statistical Physics Viewpoint: Gibbs Measures

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**Theorem [Fernique, Gayral and Sablik]**

Denote $T$ the number of cycles around a vertex.

Consider the Gibbs measure $\mu_\beta(\omega) = \frac{1}{Z} e^{-\beta E(\omega)}$, with $E$ the number of forbidden interfaces in $\omega$.

We have $\mathbb{E}_\beta[T] \leq \exp \left( \frac{9e^{2\beta}}{2(e^{\beta}+2)^2} \times \frac{1}{(e^{\beta}-2)^2} \right) - 1 < \infty$ when $\beta > \ln(2)$. 
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Figure 5: Dimer tiling on the triangle lattice, with forbidden interfaces.
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• Statistical Physics Viewpoint: Gibbs Measures

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• Information Theory Viewpoint: Bernoulli Noise
Moving onto Noisy Tilings
**Clair-Obscur Framework**

- Inject $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}} = \mathcal{A} \times \{0, 1\}$.
- Identify $\mathcal{F} \cong \widetilde{\mathcal{F}} = \mathcal{F} \times \{0\}$.
- Denote $\mathcal{M}^{B}_{\widetilde{\mathcal{F}}}(\varepsilon) \subseteq \mathcal{M}_{\widetilde{\mathcal{F}}}$ the measures with $B(\varepsilon) \otimes \mathbb{Z}^{d}$ Bernoulli noise.
- The set $\mathcal{M}^{B}_{\widetilde{\mathcal{F}}}(\varepsilon)$ is weak-* closed, and $\bigcap_{\varepsilon > 0} \mathcal{M}^{B}_{\widetilde{\mathcal{F}}}(\varepsilon) \approx \mathcal{M}_{\mathcal{F}}$.

![Figure 6: Chequerboard,](image-url)
Clair-Obscur Framework

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**Figure 6:** Chequerboard, now with obscured cells.
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Clair-Obscur Framework

- Inject $A \hookrightarrow \tilde{A} = A \times \{0, 1\}$.
- Identify $F \cong \tilde{F} = F \times \{0\}$.
- Denote $\tilde{M}^B_F(\varepsilon) \subset M_{\tilde{F}}$ the measures with $B(\varepsilon) \otimes \mathbb{Z}^d$ Bernoulli noise.
- The set $\tilde{M}^B_F(\varepsilon)$ is weak-* closed, and $\bigcap_{\varepsilon > 0} \tilde{M}^B_F(\varepsilon) \approx M_F$.

Figure 6: Chequerboard, now with obscured cells.

Reminder (Weak-* Convergence)

We say that $\mu_n \rightarrow^* \mu$ when $\mu_n([w]) \rightarrow \mu([w])$ for any finite pattern $w$. 
Figure 7: Frequency of differences between $x$ and $y$.

Finite Hamming distance:

$$d_{13 \times 8}(x, y) = \frac{33}{13 \times 8}$$
### Besicovitch Distance

**Figure 7:** Frequency of differences between $x$ and $y$.

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**Besicovitch Distance**

**Finite Hamming distance:**
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Hamming-Besicovitch pseudo-distance:

\[ d_H = \limsup_{n \to \infty} d_{n \times n} \]
### Besicovitch Distance

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Besicovitch distance on $\sigma$-invariant measures:

$$d_B(\mu, \nu) := \inf_{\lambda \text{ a coupling}} \int d_H(x, y) d\lambda(x, y) = \inf_{\lambda \text{ a coupling}} \lambda([x_0 \neq y_0])$$
Stability

Definition (Speed of Stability)

Let \( f \) s.t. \( \lim_{x \to 0^+} f(x) = 0 \). The SFT \( \Omega_\mathcal{F} \) is \( f \)-stable for \( d_B \) on Bernoulli noises if:

\[
\forall \varepsilon > 0, \quad \sup_{\lambda \in \mathcal{M}_\mathcal{F}^B(\varepsilon)} d_B (\pi_1^*(\lambda), \mathcal{M}_\mathcal{F}) \leq f(\varepsilon).
\]

Informally, a generic \( \varepsilon \)-noisy configuration will be at distance \( f(\varepsilon) \) of a tiling in \( \Omega_\mathcal{F} \).
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For the Diluted Domino tileset:

Figure 8:
Stability

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**Figure 8:** Around obscured cells, we clear the neighbourhood, and obtain a valid tiling.
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Figure 8: Around obscured cells, we clear the neighbourhood, and obtain a valid tiling.

Hence, this example is \( 5\varepsilon \)-stable.
Theorem [Gayral and Sablik, 2021, Corollary 3.15]

Let $f : \Omega_F \rightarrow \Omega_{F'}$ be a conjugacy, a bi-continuous bijection, such that, for any $k \in \mathbb{Z}^d$, we have $\sigma_k \circ f = f \circ \sigma_k$.

Then $\Omega_F$ is stable iff $\Omega_{F'}$ is. In other words, stability is a conjugacy invariant.
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What kind of (in)stability results can we expect from typical SFTs?
Conjugacy Invariance

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What kind of (in)stability results can we expect from typical SFTs?

A fixed-point argument [Durand et al., 2012] already gave a stable aperiodic example.
Stability for Periodic Tilings
1D Classification of the Stability

Figure 9: The noisy configuration is at Hamming distance $\frac{1}{2}$ of the clear $(\times o \circ o)^\infty$ ones.
1D Classification of the Stability

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Theorem [Gayral and Sablik, 2021, Theorem 4.8 and Theorem 4.9]
Consider $\Omega_F$ a 1D SFT. Then $\Omega_F$ is (linearly) stable on Bernoulli noises iff it is mixing.
Most notably, $p$-periodic SFTs (with $p \geq 2$) are unstable.
A SFT $\Omega_{\mathcal{F}}$ is (strongly) periodic if there exists an integer $N$ such that any configuration is invariant for any translation in $(N\mathbb{Z})^d$.

**Theorem [Gayral and Sablik, 2021, Theorem 5.7]**

Consider $\Omega_{\mathcal{F}}$ a 2D+ periodic SFT.

Then $\Omega_{\mathcal{F}}$ is $f$-stable on Bernoulli noises, with linear speed $f(\varepsilon) = 2C_d^{c(\mathcal{F})}\varepsilon$. 


**Lemma [Gayral and Sablik, 2021, Lemma 5.3]**

Consider a 2D+ periodic SFT $\Omega_F$.

There exists $c(F) \geq \lceil \frac{N}{2} \rceil$ such that, for any connected cell window $I \subset \mathbb{Z}^d$, if $w \in \mathcal{A}^{I+B_c}$ is locally admissible, then $w|_I$ is globally admissible.

**Figure 10:** Here, the whole domain contains no forbidden pattern, but only the blue zone is guaranteed to be the restriction of an actual configuration.
Consider $\varphi_n(b)_x = \max_{\|y-x\|_\infty \leq n} b_y$ for $b \in \{0, 1\}^{\mathbb{Z}^d}$.

Starting from a site percolation $\nu$, we obtain the $n$-thickened percolation $\varphi_n^*(\nu)$.

![Figure 11: Illustration of the mapping $\varphi_1$.](image)

**Proposition [Gayral and Sablik, 2021, Proposition 5.6]**

Consider $I \subset \mathbb{Z}^d$ the random infinite component of the $n$-thickened $B(\varepsilon)^{\otimes \mathbb{Z}^d}$-percolation.

Then $C^d_n = 48(2n + 1)^d$ is such that $\mathbb{P}(0 \notin I) \leq C^d_n \times \varepsilon$. 
Stability for Aperiodic Tilings

The Robinson Tiling
The (Enhanced) Robinson Tiling

Figure 12: Tileset and hierarchical structure of the Robinson tiling,
The (Enhanced) Robinson Tiling

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The (Enhanced) Robinson Tiling

Figure 12: Tileset and hierarchical structure of the Robinson tiling, with strengthened local rules.
High-Density Quasi-Periodic Structure

Figure 13: The density of the grid around $N$-macro-tiles goes to 0 as $N \to \infty$. 
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Stability for Aperiodic Tilings

Aperiodic Stability
Reconstruction Function for the Enhanced Tiling

**Proposition [Gayral and Sablik, 2021, Proposition 7.7]**

For any scale $N \geq 2$, the constant $C_N = 2^N - 1$ is such that for any integer $n$ and any clear locally admissible pattern $w$ on $B_{n+C_N}$, $w|_{B_n}$ is almost globally admissible, in the sense that up to a low-density grid, $w|_{B_n}$ is made of well-aligned and well-oriented $N$-macro-tiles.

---

**Figure 14:** Family of well-aligned and well-oriented tiles.
Non-linear Polynomial Stability

Theorem [Gayral and Sablik, 2021, Proposition 7.8 and Theorem 7.9]

For any $\varepsilon > 0$, any scale $N$, and any measure $\mu = \pi_1^*(\lambda)$ with $\lambda \in \widetilde{\mathcal{M}}_F^B(\varepsilon)$:

$$d_B(\mu, \mathcal{M}_F) \leq 96 \left(2^{N+2} + 1\right)^2 \varepsilon + \frac{1}{2^{N-1}}.$$  

Hence, the SFT is $f$-stable with $f(\varepsilon) = 48\sqrt[3]{6\varepsilon}$.

Could we obtain faster bounds for an aperiodic tiling?
Stability for Aperiodic Tilings

Aperiodic Unstability
A Two-Coloured Robinson Tiling

Figure 15: Two-coloured Robinson structure.
Unstable Colour Flips

Proposition [Gayral, 2021, Proposition 1]

The SFT $\Omega_{RB}$ is unstable.

More precisely, for any $\varepsilon > 0$, we have $\mu \in \mathcal{M}^{B}_{RB}(\varepsilon)$ such that $d_{B}(\mu, \mathcal{M}_{RB}) \geq \frac{1}{8}$.

**Figure 16:** The Red-Black alternating structure allows for colour flips.
Closing Thoughts
Stability for the Domino Tiling?

- For the Diluted Domino tileset:

![Figure 17:](image)

Figure 17:
Stability for the Domino Tiling?

- For the Diluted Domino tileset:

![Figure 17: Around obscured cells,](image-url)
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Figure 17: Around obscured cells, we clear the neighbourhood,
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Stability for the Domino Tiling?

• For the Diluted Domino tileset:

**Figure 17:** Around obscured cells, we clear the neighbourhood, and obtain a valid tiling.

• What about the Dense Domino phase, without the \(\square\) tile?

**Figure 18:** To pair the obscured tile as a domino, we must break another domino.
Other Open Questions

- Can we have Besicovitch-stability for Gibbs measures as the temperature goes to 0?

- Can we link stability with dynamical properties of the SFTs? (e.g. block gluing, unique ergodicity, etc.)

- Can we generalise some results to other groups $G$ with well-behaved percolations?
Léo Gayral and Mathieu Sablik. 
**On the Besicovitch-stability of noisy random tilings.**

Bruno Durand, Andrei Romashchenko, and Alexander Shen. 
**Fixed-point tile sets and their applications.**

Léo Gayral. 
**The Besicovitch-stability of noisy tilings is undecidable.**
hal.archives-ouvertes.fr/hal-03233596, 2021.
THE END OF PRESENTATION
ONE MORE SLIDE:
Thank you.
Undecidability of the Stability

$\Sigma_1$-Hardness of the Problem

Climbing the Arithmetical Hierarchy

Finding an Upper Bound
Undecidability of the Stability

$\Sigma_1$-Hardness of the Problem
Consider a Turing machine \((Q, \Gamma, I, F, \delta)\) and define the following Wang tiles:

- For any letter \(a \in \Gamma\) and any state \(q \in Q\):
- For any letter \(a \in \Gamma\) and initial state \(q \in I\):
- For any letter \(a \in \Gamma\) and final state \(q \in F\):
- For any transition \(\delta(a, q) = (b, q', L)\):
- For any transition \(\delta(a, q) = (b, q', R)\):
Figure 19: The free black tiles encode the diagram, the grey ones are communication channels.
**Figure 20:** The tiling uses an enhanced Robinson structure. It starts with Black bumpy tiles, alternates between Red and Black, then may transition to an unstable Blue-Green regime.
Transition from the Red-Black to the Blue-Green Phase

Figure 21: The flow layer appears \textit{iff} there is a final state, and propagates the colour on the border.
Theorem [Gayral, 2021, Theorem 1]

Denote $\mathcal{F}_T$ the SFT that embeds the Turing machine $T$ into a Robinson tiling.

Then $\mathcal{F}_T$ is stable (for $d_B$ on the class $\mathcal{B}$) iff $T$ does not halt on the empty input.

In the stable case, $\mathcal{F}_T$ is polynomially stable.

Because the halting problem is $\Sigma_1$-hard, so is the question of unstability.

Theorem

\textit{Stability is $\Pi_1$-hard.}
Figure 22: In a $2N$-macro-tile, only $O(12^N)$ tiles out of $16^N$ are ignored. This gives a $16^N \times C_\varepsilon + \left(\frac{3}{4}\right)^N$ bound on $d_B$ at scale $2N$. 
Idea for the Stable Case: All $N$-Macro-Tiles Are Mostly the Same

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**Figure 22:** In a $2N$-macro-tile, only $O(12^N)$ tiles out of $16^N$ are ignored.

This gives a $16^N \times C\varepsilon + \left(\frac{3}{4}\right)^N$ bound on $d_B$ at scale $2N$. 
We can do the same Blue-Green colour flip as in our Red-Black unstable example.

**Figure 23:** The transition scale plays the role of 1-macro-tiles for the Blue-Green phase.

If the Turing machine stops in the $2N$-macro-tiles, we guarantee a $\Omega \left( \frac{1}{16^N} \right)$ density of differences between Blue and Green.
We now use the following Robinson structure:

- Encode two bits \((a, b)\) in the central arm.
- The Red-Black bit \(a\) starts as Black and then alternates, for the Turing structure.
- The Blue-Green bit \(b\) starts freely and then alternates.
- Whenever the machine stops (necessarily \(a\) is Black), \(b\) must be Blue.
Dual Construction for $\Sigma_1$-Hard Stability

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Thence:

\[
\begin{align*}
T \text{ halts} & \quad \Rightarrow \quad \text{One kind of macro-tiles only at big-enough scales} & \Rightarrow \quad \text{Stable} \\
T \text{ doesn’t stop} & \quad \Rightarrow \quad \text{Two kinds of different macro-tiles at all the scales} & \Rightarrow \quad \text{Unstable}
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**Theorem**

*Stability is $\Sigma_1$-hard.*
Undecidability of the Stability

Climbing the Arithmetical Hierarchy
Figure 24: Toeplitz encoding of the sequence of colours on the left into the Robinson hierarchy.
Figure 25: In practice, we see a finite prefix of the Toeplitz encoding as a read-only input.
Generalising the $\Sigma_1$-Hard Construction on Toeplitz Inputs

Consider a Turing machine $T$ with the alphabet $\Gamma = \Sigma \sqcup \{\#\}$. We encode a word $w \in \Sigma^*\{\#\}^*$ of length $N$ in a $(2N + 1)$-macro-tile. We have three phases in the Robinson hierarchy:

1. Decoding of the Toeplitz sequence into a word $w \in \Sigma^*$.
   \[ \Downarrow \]
   Ignition of the unstable Blue-Green bit.
   \[ \Downarrow \]

2. Partial computation of $T$ on the input $w$.
   \[ \Downarrow \]
   Freezing of the now stable Blue-Green bit.
   \[ \Downarrow \]

3. $T$ halts on $w$. 

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Stability is $\Pi_2$-Hard

We have 3 situations globally speaking:

- Infinite decoding of an input $w \in \Sigma^\mathbb{N}$: stable.
- Halting of $T$ on $w \in \Sigma^*$: stable.
- Infinite computation of $T$ on $w \in \Sigma^*$: unstable through colour flips.

We have a family of bounds of the form:

$$d_B(\mu_\epsilon, M_{\mathcal{F}_T}) \leq 16\varphi(N) \times C\epsilon + \left(\frac{3}{4}\right)^N,$$

with $\varphi(N)$ the scale at which $T$ has halted on all the inputs of length at most $N$.

If $T$ halts on all the inputs, the bound still goes to 0 as $\epsilon \to 0$, but cannot be explicit as $\varphi$ can be bigger than any computable function.

**Theorem**

$\mathcal{F}_T$ is stable iff $T$ stops on all its entries, which is a $\Pi_2$-complete problem.
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Undecidability of the Stability

Finding an Upper Bound
Could the Problem be $\Pi_2$-Complete?

Stability of $\Omega_{\mathcal{F}}$: $\forall \delta > 0, \exists \varepsilon > 0$, $\sup_{\mu \in \mathcal{M}_{\mathcal{F}}^B(\varepsilon)} d_B(\mu, \mathcal{M}_{\mathcal{F}}) \leq \delta$.

By monotonicity we can consider $\varepsilon, \delta \in \mathbb{Q}^+$. 

**Theorem**

The SFT $\Omega_{\mathcal{F}}$ is stable iff it satisfies the following formula:

$$\forall \delta \in \mathbb{Q}^+, \exists \varepsilon \in \mathbb{Q}^+, \forall \rho \in \mathbb{Q}^+, \exists \gamma \in \mathbb{Q}^+, \gamma \leq \rho, \forall (w, b) \in \overline{\mathcal{W}_{\mathcal{F}}^\varepsilon}(\gamma), \exists w_0 \in \mathcal{W}_{\mathcal{F}}(\rho), \exists (w_1, w_2) \in (\mathcal{A}^2)^U_{\psi(\rho, |\mathcal{A}^2|, d)},$$

$$\left[ d_{|\mathcal{A}|}^+ (\hat{\delta}_{w_1}, \hat{\delta}_w) < 3\rho \right] \land \left[ d_{|\mathcal{A}|}^+ (\hat{\delta}_{w_2}, \hat{\delta}_{w_0}) < 3\rho \right] \land \left[ \delta_{(w_1, w_2)}(\Delta) \leq \delta + |\mathcal{A}|^2 \rho \right],$$

with $\overline{\mathcal{W}_{\mathcal{F}}^\varepsilon}(\gamma)$ and $\mathcal{W}_{\mathcal{F}}(\rho)$ being finite computable sets, and $U_{\psi(\rho, |\mathcal{A}^2|, d)}$ a computable function.

Hence a $\Pi_4$ upper bound on the problem.