Complexité et robustesse des pavages avec perturbations aléatoires

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23/06/2023, Soutenance de thèse

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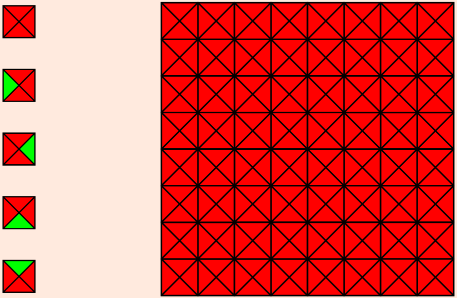
General Framework

Tilings with Local Rules

Consider the *diluted domino* tileset:



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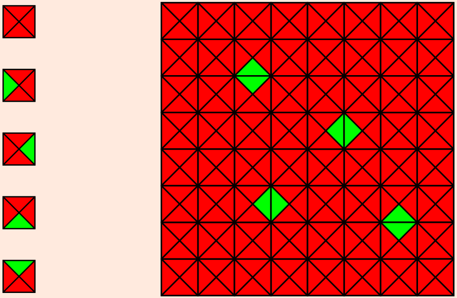

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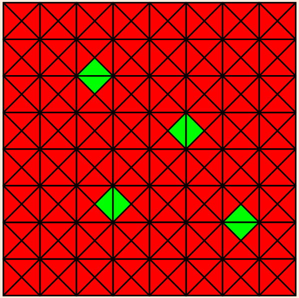

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In general, we will denote $\mathcal{A}$ the alphabet, $\mathcal{F}$ the forbidden patterns, and $X_{\mathcal{F}} \subset \Omega_{\mathcal{A}} := \mathcal{A}^{\mathbb{Z}^d}$ the subshift of admissible tilings.

Complex Structures

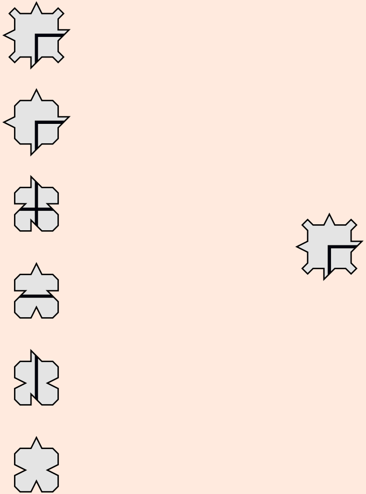

Figure 2: Hierarchical structure of the Robinson tiling.

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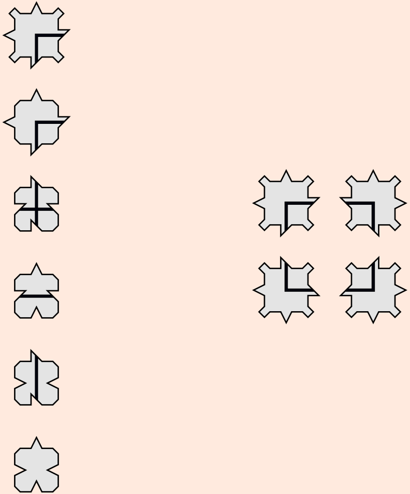

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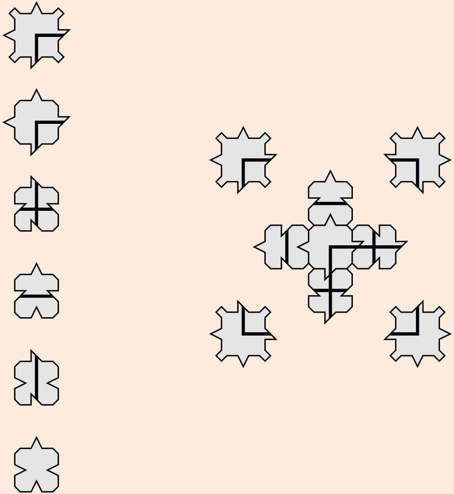

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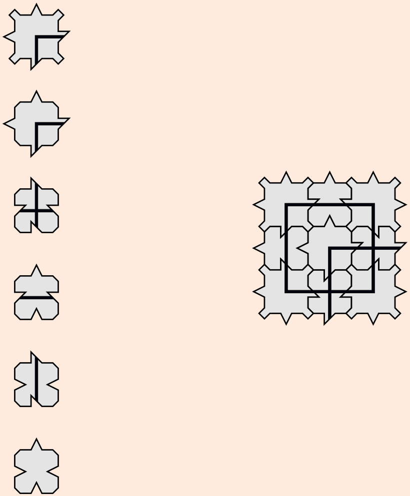

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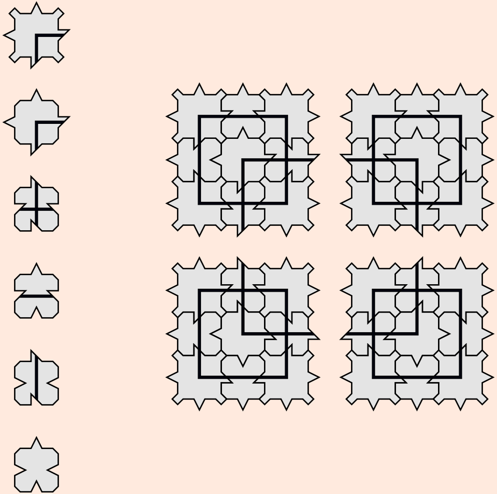

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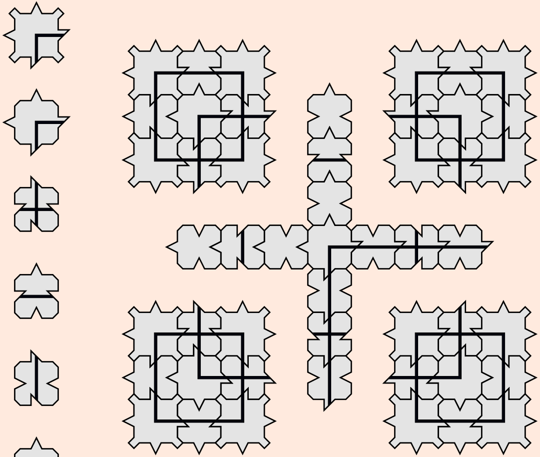

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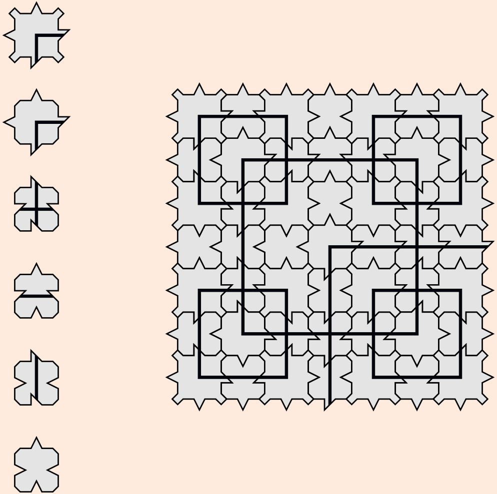

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Simulating Structures

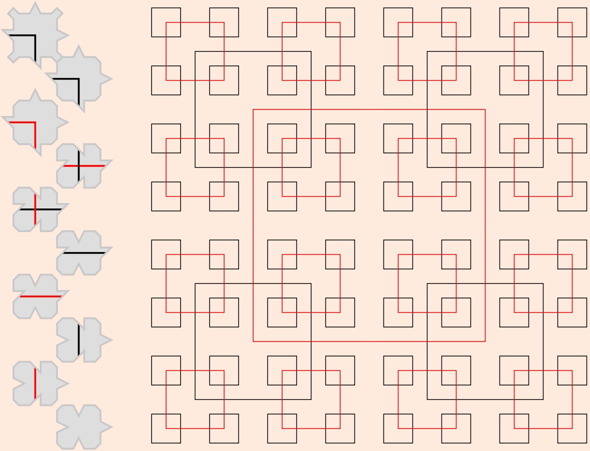

Figure 3: Bicoloured variant of the Robinson tiling, able to simulate other tilings.

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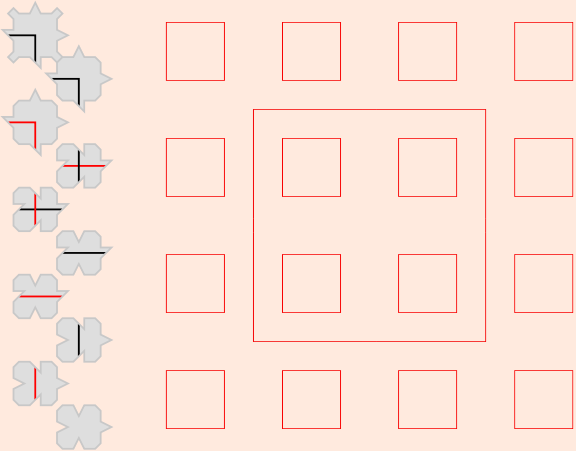

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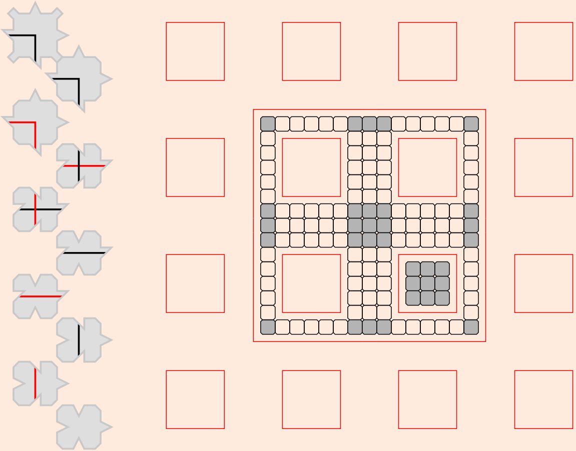

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Computability

- Computation model = computer that manipulates integers
- Decision problem = binary *yes*·*no* question
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- Halting problem: *Does this algorithm terminate?*
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- Domino problem: *Does this tileset tile the plane?*
  Halting + simulating tilesets $\Rightarrow$ Domino undecidable, Π_1-complete

Random Tilings

$\mathcal{M}_\sigma(X_\mathcal{F})$ the translation-invariant probability measures on the set of tilings $X_\mathcal{F}$.

Weak topology: $\mu_n \xrightarrow{*} \mu$ when $\mu_n([w]) \to \mu([w])$ for any finite pattern $w \in \mathcal{A}^l$.

Metrisable by computable distances $\Rightarrow$ well-suited to a computational study of measures.

Besicovitch Stability

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Bernoulli Framework

Clair-Obscur Framework

- Inject $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}} = \mathcal{A} \times \{0, 1\}$.
- Identify $\mathcal{F} \cong \widetilde{\mathcal{F}} = \mathcal{F} \times \{0\}$.
- Denote $\widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon) \subset \mathcal{M}_{\sigma}\left(X_{\widetilde{\mathcal{F}}}\right)$ the measures with $\mathcal{B}(\varepsilon)^{\otimes \mathbb{Z}^d}$ Bernoulli noise.
- The set $\widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon)$ is weak-* closed, and $\bigcap_{\varepsilon > 0} \widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon) \approx \mathcal{M}_{\sigma}\left(X_{\mathcal{F}}\right)$.

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Figure 4: Chequerboard,

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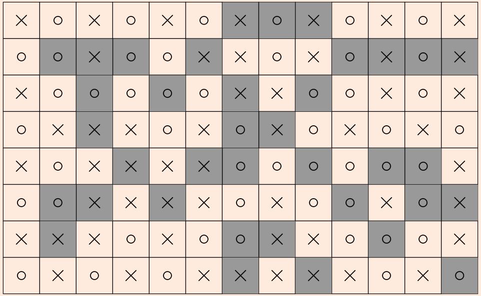

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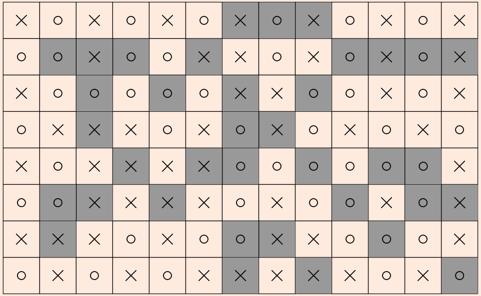

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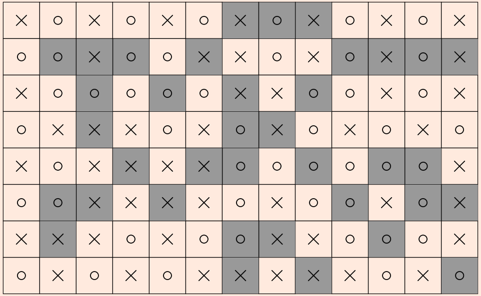

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Besicovitch Distance

Hamming-Besicovitch pseudo-distance on $\Omega_{\mathcal{A}}$: $d_H(x,y)$ the frequency of differences.

Besicovitch distance on $\mathcal{M}_\sigma\left(\Omega_{\mathcal{A}}\right)$:

$$d_B(\mu,\nu) := \inf_{\lambda \text{ a coupling}} \int d_H(x,y)\mathrm{d}\lambda(x,y) = \inf_{\lambda \text{ a coupling}} \lambda\left([x_0 \neq y_0]\right)$$

In general, the set of couplings, and thus d_B, is not computable.

Stability

Definition (Speed of Stability)

Let f s.t. $\lim_{x \to 0^+} f(x) = 0$. The SFT $X_{\mathcal{F}}$ is f-stable for d_B on Bernoulli noises if:

$$\forall \varepsilon > 0, \quad \sup_{\lambda \in \widetilde{\mathcal{M}_{\mathcal{F}}^{\mathcal{B}}}(\varepsilon)} d_B\left(\pi_1^*(\lambda), \mathcal{M}_\sigma\left(X_{\mathcal{F}}\right)\right) \leq f(\varepsilon).$$

Informally, a generic ε-noisy configuration will be at distance $f(\varepsilon)$ of a tiling in $X_{\mathcal{F}}$.

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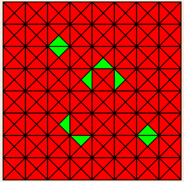

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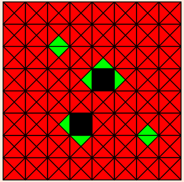

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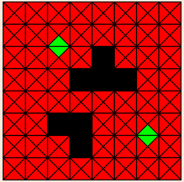

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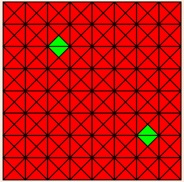

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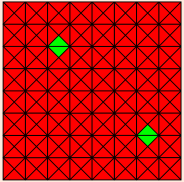

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Hence, this example is 5ε-stable.

Context

Theorem (Durand, Romashchenko, and Shen 2012)

There exists a stable aperiodic tileset, obtained using a fixed-point argument.

Informally, the bound on the speed of convergence is $\frac{1}{\sqrt{\ln(1/\varepsilon)}}$.

Theorem (Miękisz 1997)

There exists a variant of the Robinson tiling that is Besicovitch-stable, when considering the associated family of Gibbs measures.

One-Dimensional Classification

Proposition

Let $X_{\mathcal{F}}$ be a one-dimensional SFT.

Then $X_{\mathcal{F}}$ is (linearly) stable iff it is mixing.

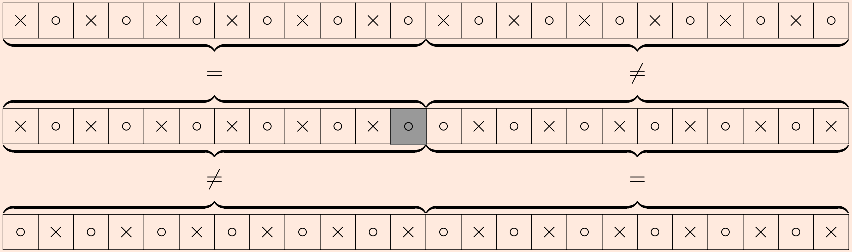

Figure 6: The noisy configuration is at Hamming distance $\frac{1}{2}$ of the clear $(\times\circ\times\circ)^{\infty}$ ones.

Besicovitch Stability

Stable Structures in Higher Dimensions

Periodic Tilings in Higher Dimensions

An SFT $X_{\mathcal{F}}$ is (strongly) periodic if there exists an integer N such that any configuration is invariant for any translation in $(N\mathbb{Z})^d$.

Theorem

Consider $X_{\mathcal{F}}$ a 2D+ periodic SFT.

Then $X_{\mathcal{F}}$ is f-stable on Bernoulli noises, with linear speed $f(\varepsilon) = 2C_{c(\mathcal{F})}^{d}\varepsilon$.

Reconstruction Function

Lemma

Consider a 2D+ periodic SFT $X_{\mathcal{F}}$.

There exists $c(\mathcal{F}) \geq \lceil \frac{N}{2} \rceil$ such that, for any connected cell window $I \subset \mathbb{Z}^d$, if $w \in \mathcal{A}^{I+B_c}$ is locally admissible, then $w|_I$ is globally admissible.

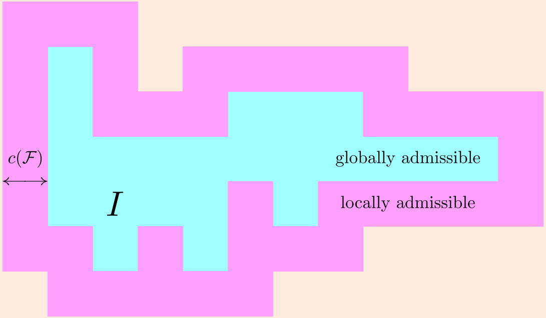

Figure 7: Here, the whole domain contains no forbidden pattern, but only the blue zone is guaranteed to be the restriction of an actual configuration.

Thickened Percolation

Consider $\varphi_n(b)_x = \max_{\|y-x\|_\infty \leq n} b_y$ for $b \in \{0,1\}^{\mathbb{Z}^d}$.

Starting from a site percolation ν, we obtain the n-thickened percolation $\varphi_n^*(\nu)$.

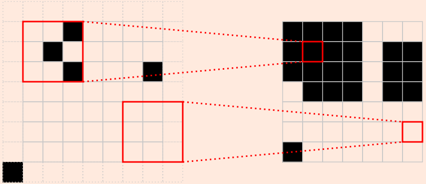

Figure 8: Illustration of the mapping φ_1.

Proposition

Consider $I \subset \mathbb{Z}^d$ the random infinite component of the n-thickened $\mathcal{B}(\varepsilon)^{\otimes \mathbb{Z}^d}$-percolation.

Then $C_n^d = 48(2n+1)^d$ is such that $\mathbb{P}(0 \notin I) \leq C_n^d \times \varepsilon$.

High-Density Quasi-Periodic Structure in the (Enhanced) Robinson Tiling

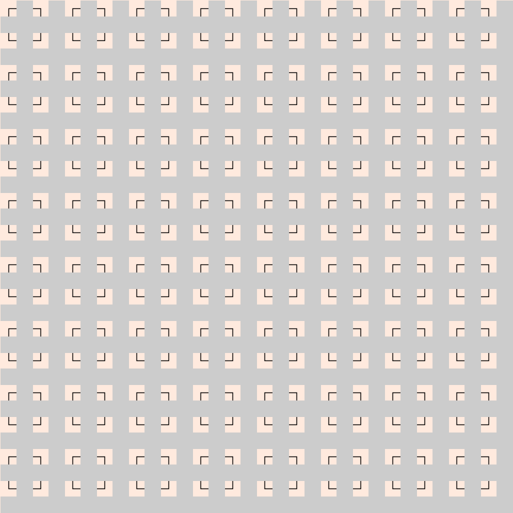

Figure 9: The density of the grid around N-macro-tiles goes to 0 as $N \to \infty$.

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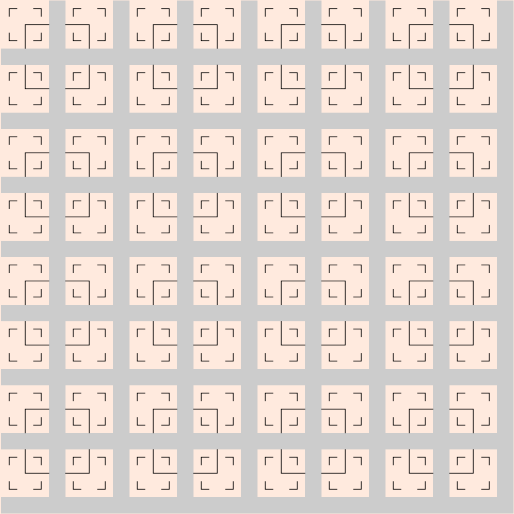

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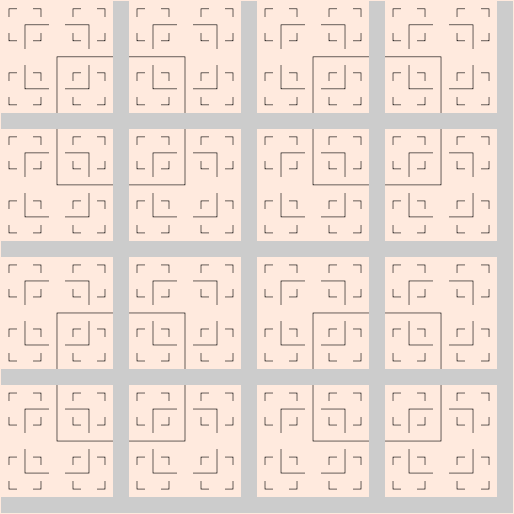

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Non-linear Polynomial Stability

Theorem

Let X_{ER} be the enhanced Robinson SFT.

For any $\varepsilon > 0$, any scale N, and any measure $\mu = \pi_1^*(\lambda)$ with $\lambda \in \widetilde{\mathcal{M}_{ER}^{\mathcal{B}}}(\varepsilon)$:

$$d_B\left(\mu, \mathcal{M}_\sigma\left(X_{ER}\right)\right) \leq 96\left(2^{N+2}+1\right)^2 \varepsilon + \frac{1}{2^{N-1}}.$$

Hence, the SFT is f-stable with $f(\varepsilon) = 48\sqrt[3]{6\varepsilon}$.

Could we obtain faster bounds for an aperiodic tiling?

Unstable Colour Flips

Proposition

The Red-Black Robinson SFT X_{RB} is unstable.

More precisely, for any $\varepsilon > 0$, we have $\mu \in \mathcal{M}^{\mathcal{B}}_{RB}(\varepsilon)$ such that $d_B\left(\mu, \mathcal{M}_\sigma\left(X_{RB}\right)\right) \geq \frac{1}{8}$.

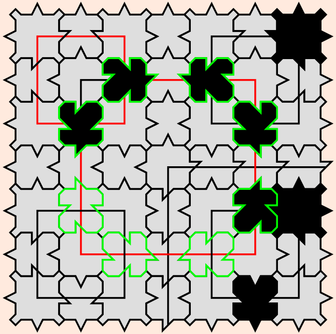


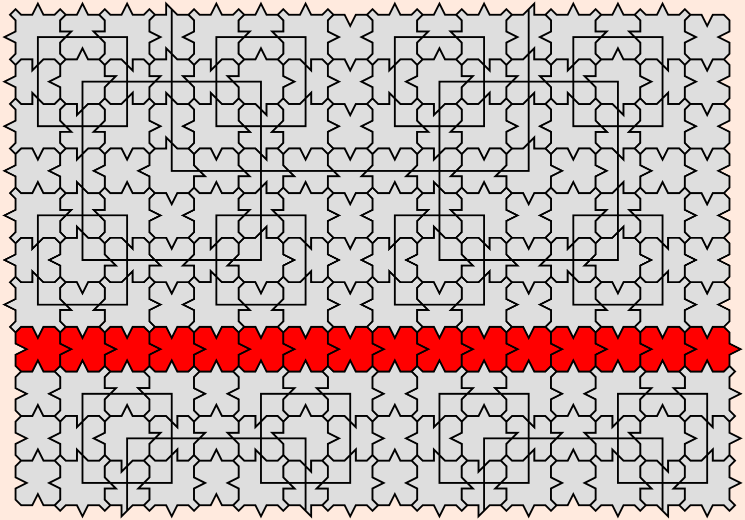
Figure 10: The Red-Black alternating structure allows for colour flips.

Undecidability of Stability

Theorem

Deciding whether $\mathcal{F}$ induces a Besicovitch-stable SFT is Π_2-hard, and at most Π_4 in the arithmetical hierarchy.

Gallery



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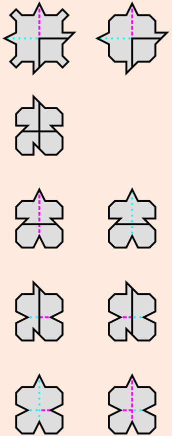

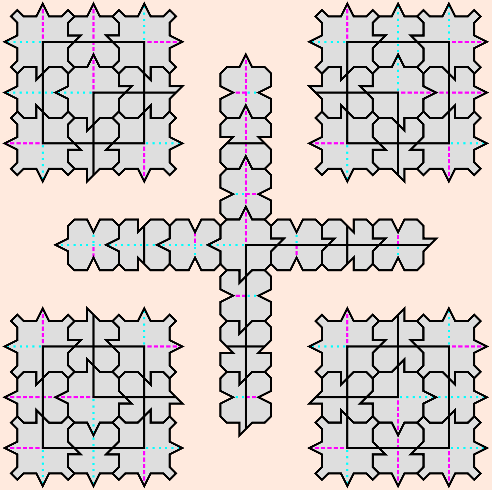

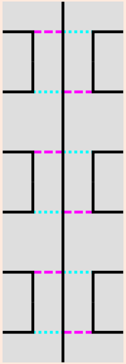

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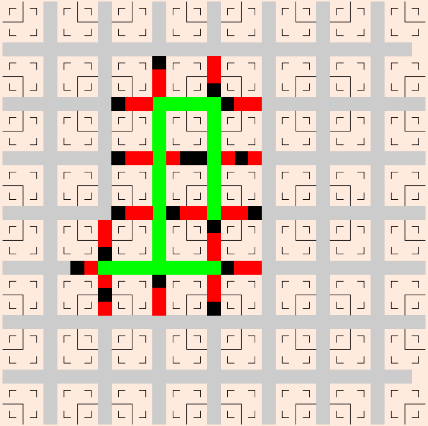

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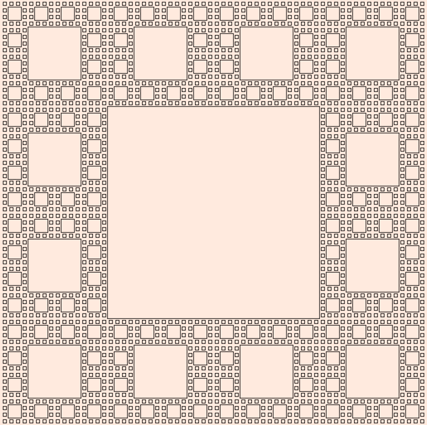

Gibbs Measures and Chaos

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Thermodynamic Framework

Gibbs Measures on Finite Spaces

- Ω a *finite* set of *states*.
- $E : \Omega \to \mathbb{R}^+$ an *energy* function.
- β the inverse temperature.

Theorem (Variational Principle)

The distribution $\mu_\beta(\omega) \propto \exp(-\beta E(\omega))$ *is the only maximiser of* $\mu \mapsto H(\mu) - \beta\mu(E)$*, with* $H(\mu) := \sum -\log_2(\mu(\omega))\mu(\omega)$ *the entropy.*

We call μ_β *a Gibbs measure.*

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We call μ_β a Gibbs measure.

- At high temperatures, as $\beta \to 0$, we converge to the uniform distribution $\mathcal{U}(\Omega)$, that maximises H.
- At low temperatures, as $\beta \to \infty$, we converge to the uniform distribution $\mathcal{U}\left(\Omega^*\right)$, that maximises H among measures of minimal energy, supported by $\Omega^* := \arg\min(E)$.

Invariant Gibbs Measures on Lattice Models

- Now, $\Omega_{\mathcal{A}} := \mathcal{A}^{\mathbb{Z}^d}$ is the phase space.
- $\varphi : \Omega_{\mathcal{A}} \to \mathbb{R}^+$ is a continuous potential, the contribution of $0 \in \mathbb{Z}^d$ to the energy.

Definition (Pressure Function)

Define the pressure $p_\mu(\beta) := h(\mu) - \beta\mu(\varphi)$,
with $h(\mu) := \lim \frac{1}{n^d} H\left(\mu_{[\![0,n-1]\!]^d}\right)$ the entropy per site.

Let $\mathcal{G}_\sigma(\beta) := \arg\max_{\mu \in \mathcal{M}_\sigma} p_\mu(\beta)$ the set of Gibbs measures.

- φ has finite range if it is *locally constant*, if $\varphi(\omega)$ only depends on $\omega_{[\![-r,r]\!]^d}$.

Limit Behaviour for Ground States

- We call $(\mu_\beta \in \mathcal{G}_\sigma(\beta))_{\beta>0}$ a *cooling trajectory* of the model.
- Denote $\mathcal{G}_\sigma(\infty) := \mathrm{Acc}_{\beta\to\infty}\,\mathcal{G}_\sigma(\beta)$ the set of *ground states*, of accumulation points of all the cooling trajectories.

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Lemma

Assume that $X := \left\{\omega \in \Omega_{\mathcal{A}}, \forall x \in \mathbb{Z}^d, \varphi \circ \sigma^x(\omega) = 0\right\} \neq \emptyset$.
Then $\mathcal{G}_\sigma(\infty) \subset \mathcal{M}_\sigma(X)$*, and the ground states have maximal entropy* h *in* $\mathcal{M}_\sigma(X)$.

- Measures that maximise h in $\mathcal{M}_\sigma(X)$ are not necessarily in $\mathcal{G}_\sigma(\infty)$.

What can we ask about $\mathcal{G}_\sigma(\infty)$?

Stability and Chaos

Definition (Stability)

A model is stable if all the cooling trajectories converge to the same limit.

Definition (Chaoticity)

A model is chaotic if there is no converging cooling trajectory.

Definition (Uniformity)

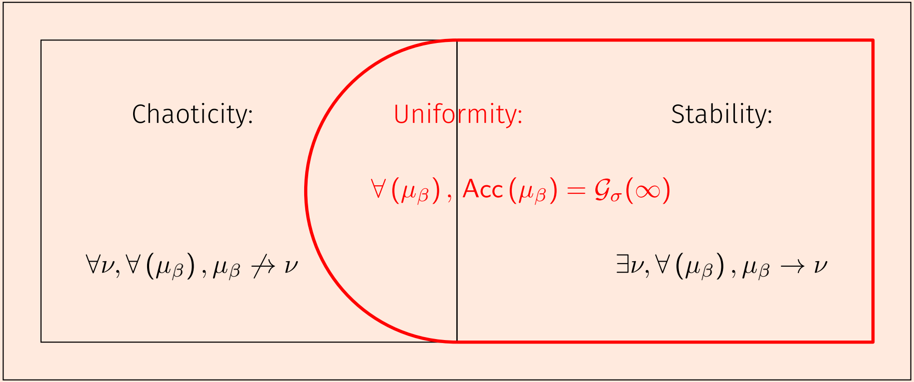
A model is uniform if all the cooling trajectories have the same accumulation set.

Recap of Behaviours

Chaoticity:	Stability:
$\forall \nu, \forall (\mu_\beta), \mu_\beta \not\to \nu$	$\exists \nu, \forall (\mu_\beta), \mu_\beta \to \nu$

Figure 11: Inventory and comparison of model behaviours.

Recap of Behaviours



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Recap of Behaviours

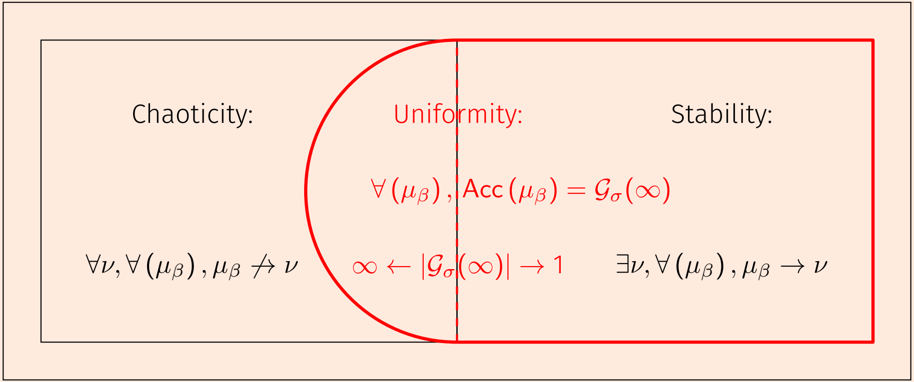

Figure 11: Inventory and comparison of model behaviours.

Current Knowledge

Lemma (Brémont 2003)

A one-dimensional finite range model induces a stable model.

Theorem (Chazottes and Hochman 2010)

There exists a one-dimensional potential inducing a chaotic model.

There exists a three-dimensional finite range potential inducing a chaotic model.

Theorem (Chazottes and Shinoda 2020; Barbieri et al. 2022)

There exists a two-dimensional finite range potential inducing a chaotic model.

Gibbs Measures and Chaos

Obstructions and Realisation

Topological Obstruction on the Accumulation Set

Proposition

For any potential φ, the set $\mathcal{G}_\sigma(\infty)$ is connected.

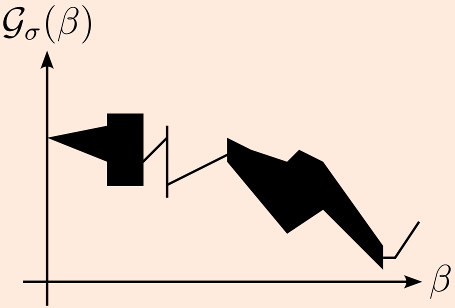

Figure 12: The graph of $\beta \mapsto \mathcal{G}_\sigma(\beta)$ is connected.

Computational Obstruction on the Accumulation Set

Forbidden patterns induce only countably many potentials (and accumulation sets).

What are we missing?

Computational Obstruction on the Accumulation Set

Forbidden patterns induce only countably many potentials (and accumulation sets).
What are we missing?

Proposition

For any computable potential φ inducing a uniform model, $\mathcal{G}_\sigma(\infty)$ is Π_2-computable.

Without the uniformity assumption, we have a Π_3 upper bound instead.

Notably, Π_2-computable sets can be characterised as limit sets of computable sequences.

General Idea for Chaoticity

We have two measures $\mu \neq \mu'$ s.t. $d(\mu, \mu') \geq r$ and:

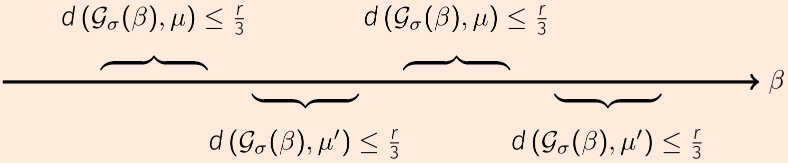

Figure 13: Alternating between mutually exclusive adherence values on non-overlapping intervals.

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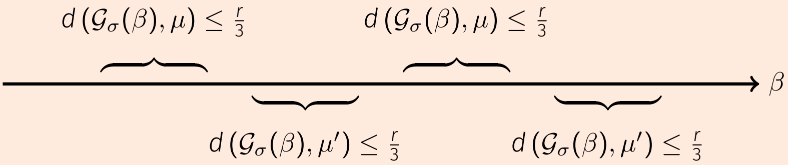

Figure 13: Alternating between mutually exclusive adherence values on non-overlapping intervals.

Thus $\mathrm{Acc}(\mu_\beta)$ intersects the disjoint neighbourhoods of both μ and μ'.

General Idea for Uniformity

We want (μ_n) and $\varepsilon_n \to 0$ s.t.:

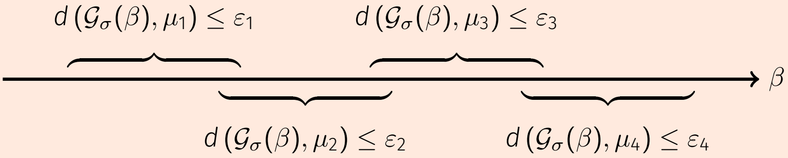

Figure 14: Contracting tube of measures with overlapping intervals.

General Idea for Uniformity

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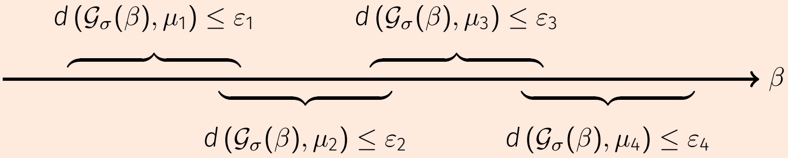


Figure 14: Contracting tube of measures with overlapping intervals.

Thus $\mathrm{Acc}(\mu_\beta) = \mathcal{G}_\sigma(\infty) = \mathrm{Acc}(\mu_n)$.

Realisation Result on the Limit Set

Theorem

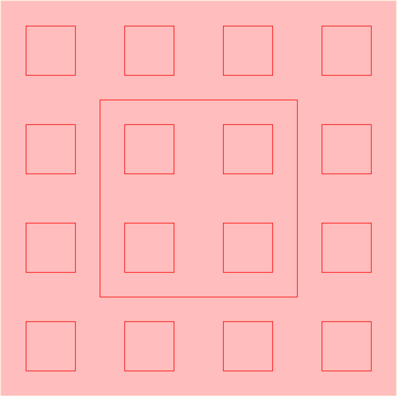
There exists a class of two-dimensional finite range potentials, inducing uniform models both stable and chaotic.

More precisely, we can realise any connected Π_2-computable compact set X as $\mathcal{G}_\sigma(\infty)$, up to a fixed computable affine homeomorphism.

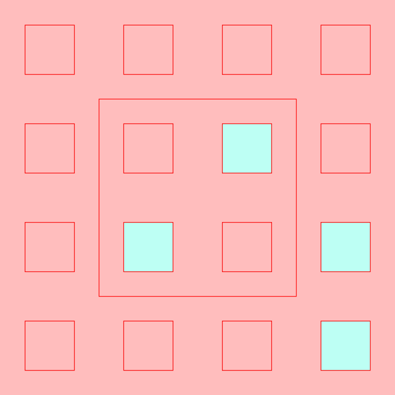
Corollary

The question of whether a computable φ induces a chaotic model is Σ_3-complete.

Gallery



Gallery



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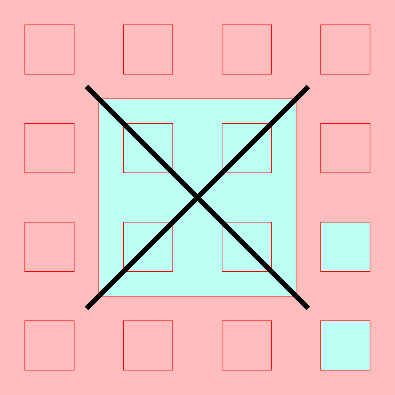

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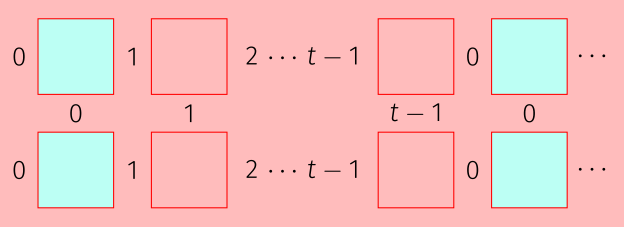

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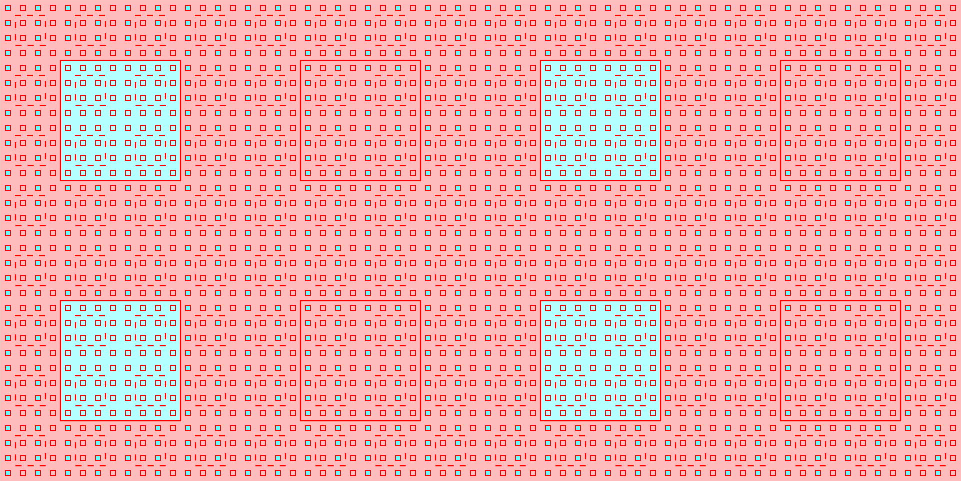

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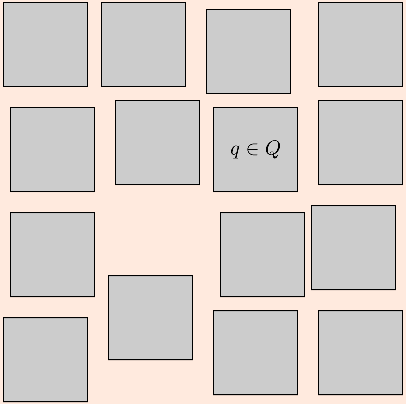

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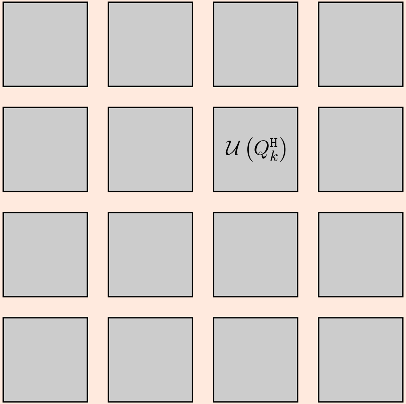

Bibliography

- Barbieri, Sebastián et al. (2022). ***Chaos in Bidimensional Models with Short-Range.*** `10.48550/arXiv.2208.10346`.
- Brémont, Julien (2003). **"Gibbs Measures at Temperature Zero".** In: *Nonlinearity* 16.2, pp. 419–426. `10.1088/0951-7715/16/2/303`.
- Chazottes, Jean-René and Michael Hochman (2010). **"On the Zero-Temperature Limit of Gibbs States".** In: *Communications in Mathematical Physics* 297.1, pp. 265–281. `10.1007/s00220-010-0997-8`.
- Chazottes, Jean-René and Mao Shinoda (2020). ***On the Absence of Zero-Temperature Limit of Equilibrium States for Finite-Range Interactions on the Lattice*** $\mathbb{Z}^2$**.** `10.48550/arXiv.2010.08998`.
- Durand, Bruno, Andrei Romashchenko, and Alexander Shen (2012). **"Fixed-point tile sets and their applications".** In: *Journal of Computer and System Sciences* 78.3, pp. 731–764. `10.1016/j.jcss.2011.11.001`.
- Miękisz, Jacek (1997). **"Stable Quasicrystalline Ground States".** In: *Journal of Statistical Physics* 88.3–4, pp. 691–711. `10.1023/B:JOSS.0000015168.25151.22`.

THE END OF PRESENTATION

ONE MORE SLIDE:

Thank you.