Research and Integration Project: Robustness and Complexity of Perturbed Discrete Structures

Léo Gayral

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1 General Overview

I started working on the question of *Robustness of Tilings Under Random Perturbations* with Mathieu Sablik in September 2020 and finished my PhD in June 2023, and my PhD dissertation can be found on my webpage. My work is at the interface between statistical physics, discrete dynamical systems, percolation theory, combinatorics, complexity theory and computable analysis.

In this context, a tiling is typically made of square tiles aligned on a grid (or more generally an integer lattice $\mathbb{Z}^d$), with "local rules" that determine which tiles are allowed next to each other, like puzzle pieces. One of the most known examples of the field is the Robinson tiling [16], which uses 32 tiles in total (including rotations and symmetries of the base tiles in Figure 1), with a hierarchical structure where the black lines on the tiles form squares that are embedded into bigger squares.

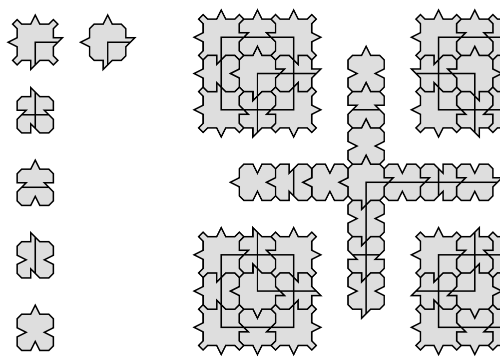

Figure 1: Base tiles and hierarchical structure of the Robinson tiling.

Random perturbations naturally correspond to the presence of a low amount of violation of the local rules, of mismatched tiles. Consequently, robustness describes the fact that tilings with small perturbations are close to tilings without, with respect to some property or in some topology.

These dynamic properties generally have strong ties with computability properties. For example, through the encoding of Turing machines within its hierarchical structure, the Robinson tiling served to prove that the question of whether a tileset *can* tile the whole grid is equivalent to the halting problem [16], hence a Σ_1-complete undecidable problem. The interactions between dynamic and computability properties of similar systems have been an active field of study in the last decades [11, 1, 12, 2, 4, 5, 13, 6, 3].

So far, I have explored two main directions, using two different kinds of noises. On one hand, with a Bernoulli noise (where each cell can violate rules independently with low probability), I focused on stability for the Besicovitch topology (*i.e.* the frequency of differences), up to the fact that this property is undecidable. This work has been released through two preprints, one of them now published in the Electronic Journal of Probability and the other in Theory of Computing Systems, and is explained in Section 2.1. On the other hand, with Gibbs measures and with the weak-$*$ topology, I focused on a notion of chaoticity of models as the temperature goes to 0. This resulted in a recent preprint, and some ideas are exposed in Section 2.2.

Then, in Section 3, I summarise some research problems at hand on which I may be most likely to work in the near future, *i.e.* the research project. Finally, in Section 4, I detail the logistics of the integration project.

2 Previous Works

2.1 First PhD Project

My first results are gathered in the article *On the Besicovitch-Stability of Noisy Random Tilings* [8]. At the time, I focused on matters relating purely to stability, with little to no computation theory involved.

In this context, we consider translational-invariant noisy measures on an SFT for which each cell can violate a rule with probability ε independently of other cells (*i.e.* a Bernoulli noise). To compare two random tilings, we use the Besicovitch distance d_B, which quantifies the frequency of differences between configurations. A system is (resp. linearly, polynomially) *stable* if the induced distance between ε-noisy measures and non-noisy ones goes to 0 as $\varepsilon \to 0$ (resp. at speed $O(\varepsilon)$, or $O(\varepsilon^r)$ for some rate $0 < r \leq 1$). We first prove that this property is a conjugacy invariant, and characterise stable one-dimensional systems as mixing SFTs. Then, we jump into the higher-dimensional case, where we use tools from percolation theory to prove that SFTs that behave periodically in every direction are linearly stable. Finally, we adapt this idea to an aperiodic (yet *quasi-periodic*) SFT:

Theorem 1 (Stability of the enhanced Robinson tiling [8, Theorem 7.9])**.** *Let $X_\mathcal{F}$ be the (locally enhanced) Robinson tiling. This SFT is polynomially stable, with a bound $O\left(\sqrt[3]{\varepsilon}\right)$ on the speed of convergence.*

The second main paper (of which some preliminary results were presented in an exploratory article [7]) focuses more on the computational aspects of the previous stability question, and is called *Arithmetical Hierarchy of the Besicovitch-Stability of Noisy Tilings* [9].

In this article, we consider the question of whether a set of forbidden patterns $\mathcal{F}$ induces a stable SFT $X_\mathcal{F}$. Historically, similar works were done in particular on the Domino Problem (whether $X_\mathcal{F} \neq \emptyset$ given $\mathcal{F}$), which is a known Σ_1-complete in the arithmetical hierarchy. Here, the class Σ_k denotes questions on an input x that are answered by a formula $\exists y_1 \forall y_2 \exists y_3 \dots \varphi(x, y_1, \dots, y_k)$ with φ a computable function (and likewise for the class Π_k, starting with $\forall y_1$ instead). A problem is Σ_k-hard if, for *any* Σ_k problem, we have a computable reduction to the initial problem. The main result of this article can be summed up as follows:

Theorem 2 (Undecidability of the Besicovitch-stability [9, Theorems 5.11 and 6.8])**.** *The question of stability of a SFT $X_\mathcal{F}$, as a function of $\mathcal{F}$, is Π_2-hard and at most Π_4 in the arithmetical hierarchy.*

It follows that this problem is undecidable, and strictly harder than the halting problem. However, we couldn't manage to improve either bounds to state an optimal completeness result.

2.2 Second PhD Project

After hitting the previous roadblocks, my attention shifted of other questions of stability, for Gibbs measures and with the weak-$*$ topology this time. This work stemmed from conversations with Siamak Taati (American University of Beirut) in summer 2021, and resulted in a recent prepublication [10]. In this context, following the lead of Chazottes and Hochman [4] a dozen years ago (with tilings of the lattice $\mathbb{Z}^3$), we studied *how* the noisy measures converge as the system cools down (*i.e.* as the amount of noise goes to 0), rather than *if* they do so.

More precisely, we are studying the question of uniform chaoticity of models of Gibbs measures $(\mathcal{G}_\sigma(\beta))_{\beta>0}$, induced by finite-range potentials $\varphi_\mathcal{F}$ associated to SFTs (for which the ground states $\mathcal{G}_\sigma(\infty)$ are invariant measures on $\Omega_\mathcal{F}$). Gibbs measures $\mu_\beta \in \mathcal{G}_\sigma(\beta)$ are those that maximise the *pressure* $p_\mu(\beta) = h(\mu) - \beta\mu(\varphi)$. A uniform model is such that *any* cooling trajectory $(\mu_\beta)_{\beta>0}$ accumulates on the whole set of ground states $\mathcal{G}_\sigma(\infty)$. Our main result can be summarised as follows:

Theorem 3 (Uniform Chaoticity)**.** *Let X be a* connected *Π_2-computable set of probability measures on $\{-1, +1\}^\mathbb{N}$. Then there exists a finite-range potential $\varphi_\mathcal{F}$, and an affine embedding of X into invariant measures on $\Omega_\mathcal{F}$, such that for* any *cooling trajectory $(\mu_\beta)_{\beta>0}$, we have the same accumulation set $Acc_{\beta\to\infty}(\mu_\beta) = \mathcal{G}_\sigma(\infty) \cong X$. This bound is tight, in that for any uniform model, the set of ground states is Π_2-computable.*

To obtain this result, we build in particular a family of zero-entropy tilings $X_\mathcal{F}$, in which the combinatorial structure forced by the tileset allows for a fine control on how slowly the entropy on finite windows goes to 0.

3 Research Project

One direction where there may be some results within reach is the study of the Besicovitch-stability (from Section 2.1) of systems of Gibbs measures, instead of ones with a Bernoulli noise. My first article [8] also triggered a conversation around how different notions of stability relate to each other, in particular after taking into account some earlier stability results by Durand, Romashchenko and Shen [5] (who proved the Besicovitch-stability of some *robust* tilings, obtaining along the way some ill-defined stronger notion of "stability of all the scales of the tiling"). A promising goal for the future may be to coin a formal definition for this stronger notion of stability, and an appropriate framework to study it.

Still related to Besicovitch-stability, but moving onto the thermodynamic formalism, it may be interesting to see whether or not stability is a conjugacy invariant in this case, given the intrinsically "non-local" definition of Gibbs measures, at odds with the local characterisation of conjugacies.

Another direction still open, which would be a natural follow-up to my most recent work, would be to realise Π_3-computable accumulation sets for non-uniform models, as we also established a Π_3 upper bound for any model induced by a *computable* potential. This would imply a profound correspondence between the dynamical behaviour of statistical physics models at low temperatures and the computational complexity of their accumulation sets.

I am also highly interested in working again on the topic of my Master's internship, which happened during the harsh context of France's lockdown. The initial motivation for this internship stems from two papers published at the same time in the early 90s, in a context of empirical / experimental mathematics, that obtain opposite conclusions on a question of phase transition in random Penrose tilings. For this internship with Thomas Fernique, I studied Gibbs measures on dimer tilings, seen as discrete surfaces, and I tried to obtain a phase transition regarding their height function. Using a Peierls argument, the low-temperature case was settled, but we still lack arguments to conclude on the high-temperature behaviour, and on a monotonicity result to justify an actual phase transition. Considering how this statistical physics model is analogous to the loop $O(n)$ model [15], deeper exchanges with the Percolation community would probably further my insight on the matter. The ultimate end-goal of this study would then be to transpose some of these results to the case of a random Penrose tiling with local rules instead.

Still in the context of phase transitions for Gibbs measures, my university granted me funding to go and meet Anthony Quas in Canada. The main motivation of this exchange was to see if his result, about controlling precisely at which temperatures you have non-uniqueness of the Gibbs measures (*i.e.* a phase transition), using a long-range one-dimensional potential, could be replicated in higher dimension with a finite-range potential. A breakthrough did not occur yet, but the question deserves more attention, in particular regarding the possibility of a potential inducing a *freezing* phase, such that Gibbs measures are actually zero-energy ground states at low-enough temperatures.

These are just some examples of the many open question I encountered during my PhD without having the time to properly study them, but my interests are broad and I would also be delighted to join a group to work on other similar problems. One of my local contacts, Pierre Guillon, notably suggested topics relating to the classification of S-adic tilings, with non-local constraints, or to self-assembling algorithms.

4 Integration Project

In its current form, the project is to be based at the I2M, where I gave a presentation last March for the Rauzy seminar and Pytheas Fogg workgroup. My two closest scientific contacts there, Guillaume Theyssier and Pierre Guillon, wrote one of the attached support letters. As underlined by Peter Haïssinsky (I2M director) in his support letter, I would not only benefit from a workspace at the I2M but also at the CIRM, where I have frequently been for conferences, and probably will keep doing so.

While Pierre Guillon and Guillaume Theyssier represent a part of the GDAC team closer to what I have already worked on during my PhD (relating to computational complexity), I also have the support of Pascal Hubert (CIRM director) and established exchanges with Nicolas Bédaride, with opportunities to explore further nearby problems relating to the thermodynamic formalism. It would be a delight to have the opportunity to exchange and collaborate with all of them.

My research topics may also be explored with researchers at the CPT, a bit closer to physics, or at the LIS, with researchers closer to computer science. This would require me to be more familiar with the local communities first, but may also allow for opportunities to create bridges between them.

For all these reasons, scientifically speaking, Marseille is one of the best environments worldwide to extend my past research further, while also being conveniently located in France, allowing me to efficiently access CNRS and Inria, where I intend to find a full-time research position in the long term.

Regarding accommodations, my basic needs are a calm workspace and flexible hours, which can be offered for no cost as long as I don't have teaching duties. On the longer term, I really need stability and structures to work properly, and on that regard, the three year length of this funding is already exceptionally rare and would be greatly appreciated.

References

[1] Eugene Asarin and Pieter Collins. Noisy Turing machines. In *ICALP 2005: Automata, Languages and Programming, 32nd International Colloquium*, volume 3580 of *Lecture Notes in Computer Science*, pages 1031–1042. Springer, 2005.

[2] Nathalie Aubrun and Mathieu Sablik. Simulation of effective subshifts by two-dimensional subshifts of finite type. *Acta Applicandae Mathematicae. An International Survey Journal on Applying Mathematics and Mathematical Applications*, 126:35–63, 2013.

[3] Paul Balister, Béla Bollobás, Robert Morris, and Paul Smith. Universality for monotone cellular automata, 2022.

[4] Jean-René Chazottes and Michael Hochman. On the zero-temperature limit of Gibbs states. *Communications in Mathematical Physics*, 297(1):265–281, 2010.

[5] Bruno Durand, Andrei Romashchenko, and Alexander Shen. Fixed-point tile sets and their applications. *Journal of Computer and System Sciences*, 78(3):731–764, 2012.

[6] Nazim Fatès, Irène Marcovici, and Siamak Taati. Self-stabilisation of cellular automata on tilings. *Fundamenta Informaticae*, 185(1):27–82, 2022.

[7] Léo Gayral. The Besicovitch-stability of noisy tilings is undecidable. hal-03233596, 2022.

[8] Léo Gayral and Mathieu Sablik. On the Besicovitch-stability of noisy random tilings. *Electronic Journal of Probability*, 28:1–38, 2023. 10.1214/23-EJP917.

[9] Léo Gayral and Mathieu Sablik. Arithmetical hierarchy of the Besicovitch-stability of noisy tilings. *Theory of Computing Systems*, 2023. 10.1007/s00224-023-10142-y.

[10] Léo Gayral, Mathieu Sablik, and Siamak Taati. Characterisation of the set of ground states of uniformly chaotic finite-range lattice models. arXiv:2302.07326, 2023.

[11] Peter Gács. Reliable cellular automata with self-organization. *Journal of Statistical Physics*, 103(1-2):45–267, 2001.

[12] Olle Häggström. Computability of percolation thresholds. In *In and out of equilibrium. 2*, volume 60 of *Progr. Probab.*, pages 321–329. Birkhäuser, Basel, 2008.

[13] Emmanuel Jeandel and Pascal Vanier. Hardness of conjugacy, embedding and factorization of multidimensional subshifts. *Journal of Computer and System Sciences*, 81(8):1648–1664, 2015.

[14] Jacek Miękisz. A microscopic model with quasicrystalline properties. *Journal of Statistical Physics*, 58(5-6):1137–1149, 1990.

[15] Ron Peled and Yinon Spinka. Lectures on the spin and loop $O(n)$ models. In *Sojourns in Probability Theory and Statistical Physics - I*, pages 246–320. Springer, 2019.

[16] Raphael Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones mathematicae*, 12:177–209, 1971.